Differential Geometry III: Exercises

University of Regensburg, Winter term 2021/22
Prof. Dr. Bernd Ammann, Jonathan Glöckle
Please hand in the exercises until **Friday, December 17**

Exercises Sheet no. 8

1. Exercise (4 points).
Let (M, g) be a semi-Riemannian manifold. Let $f \in C^\infty(M)$ and $p \in M$ be a stationary point of f. Show the following:

a) The Hessian of f in p, defined by $\mathrm{Hess}_p f = \nabla \mathrm{d} f_{|p} \in \bigodot^2 T_p^* M$, is independent of the chosen connection (or the semi-Riemannian metric, if we consider Levi-Civita connections).

b) If p is a local maximum, then $\mathrm{Hess}_p f$ is negative semi-definite.

c) If $\mathrm{Hess}_p f$ is negative definite, then p is a local maximum of f.

Hint: Recall that $\nabla \mathrm{d} f(X, Y) = \partial_X \partial_Y f - \partial_{\nabla_X Y} f$ *for* $X, Y \in \Gamma(TM)$ *by definition.*

2. Exercise (4 points).
Let (M, g) be an n-dimensional Riemannian manifold. Suppose that $u : M \to (0, \infty)$ is a smooth solution of the equation

$$L^g u = c u^{(n+2)/(n-2)}.$$

a) Assume that $\tilde{g} = \phi^{4/(n-2)} g$ for some smooth function $\phi : M \to (0, \infty)$. Show that $\tilde{u} := \phi^{-1} u$ satisfies

$$L^{\tilde{g}} \tilde{u} = c \tilde{u}^{(n+2)/(n-2)}.$$

Furthermore prove that for $p = 2n/(n-2)$ we have

$$\|u\|_{L^p(M,g)} = \|\tilde{u}\|_{L^p(M,\tilde{g})} \in [0, \infty].$$

b) Let $\Phi \in \mathrm{Conf}(M, g)$. Define the function $\phi : M \to (0, \infty)$ by requiring $\Phi^* g = \phi^{4/(n-2)} g$. Then show that $\tilde{u} := \phi \Phi^* u = \phi \cdot (u \circ \Phi)$ is again a solution of

$$L^g \tilde{u} = c \tilde{u}^{(n+2)/(n-2)}.$$

Furthermore prove that for $p = 2n/(n-2)$ we have

$$\|u\|_{L^p(M,g)} = \|\tilde{u}\|_{L^p(M,g)} \in [0, \infty].$$

Hint: Both parts can be shown similarly, using the relation between L^g *and* $L^{\tilde{g}}$ *mentioned in the lecture. One also may use (a) to prove (b); in fact both parts are even equivalent.*

3. Exercise (4 points).
Let g_{sph} be the standard metric on S^n and g_{eucl} the Euclidean metric on $\mathbb{R}^{n-1}$. Verify that the stereographic projection $\sigma : (S^n \smallsetminus \{e_0\}, g_{\mathrm{sph}}) \to (\mathbb{R}^{n-1}, g_{\mathrm{eucl}})$ is conformal by explicitly calculating

$$(\sigma^{-1})^* g_{\mathrm{sph}|x} = \frac{4}{(\|x\|^2 + 1)^2} g_{\mathrm{eucl}|x}.$$

Hint: In case you do not see how to make this calculation, reading in the Yamabe script by Ammann and Bär might help.