Differential Geometry III: Exercises

University of Regensburg, Winter term 2021/22
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Please hand in the exercises until **Friday, December 10**

Exercises Sheet no. 7

1. Exercise (4 points).
Let (M, g) be a compact n-dimensional manifold. For a continuous function $w : M \to \mathbb{R}$ and some $p \in \mathbb{R}$, $p \geq 2$, we define the functional

$$\mathcal{F}(u) = \frac{\int_M (|\nabla u|^2 + w u^2)\,\mathrm{dvol}}{\|u\|_{L^p}^2}.$$

Here, as usual, the L^p-norm is defined by $\|u\|_{L^p} = \left(\int_M |u|^p\,\mathrm{dvol}\right)^{\frac{1}{p}}$. Show that for $u \in C^\infty(M)$, the following are equivalent:

(i) For all $v \in C^\infty(M)$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{F}(u + tv) = 0.$$

(ii) There is a constant $c \in \mathbb{R}$ with

$$\Delta u + wu = c|u|^{p-2}u.$$

Additionally, if one and hence both of the equivalent conditions are satisfied, express the constant c in terms of $\mathcal{F}(u)$ and $\|u\|_{L^p}$.

2. Exercise (4 points).
Let $\mathcal{L} := \{X \in \mathbb{R}^{n+1,1} \setminus \{0\} \mid \langle\!\langle X, X \rangle\!\rangle = 0\}$ be the set of light-like vectors in Minkowski space.

a) Show that $\mathcal{L}$ is a smooth (but not semi-Riemannian!) submanifold of $\mathbb{R}^{n+1,1}$ and $\iota : S^n \to \mathcal{L}$, $x \mapsto (1, x)$ is an embedding of smooth manifolds[1]. Prove furthermore that $S^n \xrightarrow{\iota} \mathcal{L} \to \mathcal{L}/\mathbb{R}^*$ is a bijection, giving rise to a smooth map $\pi : \mathcal{L} \to \mathcal{L}/\mathbb{R}^* \cong S^n$.

b) Let $x \in \mathcal{L}$. A space-like vector $v \in T_x\mathcal{L}$ defines an oriented plane $E_v = \mathrm{span}(x, v)$ in $\mathbb{R}^{n+1,1}$ ((x, v) shall be positively oriented in E_v, say). Given two oriented planes E and F such that $E = E_v$ and $F = E_w$ for some space-like $v, w \in T_x\mathcal{L}$, show that their enclosed angle

$$\angle(E, F) = \arccos \frac{\langle\!\langle v, w \rangle\!\rangle}{\sqrt{\langle\!\langle v, v \rangle\!\rangle}\sqrt{\langle\!\langle w, w \rangle\!\rangle}} \in [0, \pi]$$

is well-defined (i. e. independent of the choices of v and w). Verify additionally that $\angle(E, F) = \angle(\mathrm{d}_x\pi(v), \mathrm{d}_x\pi(w))$, where the latter is the angle between two vectors on the standard sphere (S^n, g).

c) Let $A \in O(n + 1, 1)$. Show that A restricts to a diffeomorphism $A : \mathcal{L} \to \mathcal{L}$, that there is a unique map $\widetilde{A}$ making the diagram

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{A} & \mathcal{L} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ S^n & \xrightarrow{\widetilde{A}} & S^n \end{array}$$

commute and that $\widetilde{A}$ is a conformal diffeomorphism.

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[1] including being a homeomorphism onto its image

3. Exercise (4 points).
Let $x^1 \colon S^n \to \mathbb{R}$ be the restriction of the function $\mathbb{R}^{n+1} \ni x = (x_1, \ldots, x_{n+1}) \to x_1$ and let $X = \operatorname{grad} x^1 \in \Gamma(TS^n)$, where the gradient is taken with respect to the standard metric g on S^n.

a) Show that X is a conformal Killing field, more precisely,

$$\mathcal{L}_X g = -2x^1 \cdot g.$$

Hint no.1: Verify first $X_p = e_1 - x^1(p) \cdot p \in T_pS^n \subset \mathbb{R}^{n+1}$.
Hint no. 2: if $\overline{\nabla}$ *is the connection on* $\mathbb{R}^{n+1}$ *and if* $\overline{\operatorname{grad}}$ *is the gradient of* $\mathbb{R}^{n+1}$, *then the relations* $\overline{\operatorname{grad}}\, x^1 = e_1$ *and* $\overline{\nabla} e_1 = 0$ *might be helpful.*

b) Conclude that $\Delta x^1 = n x^1$.