Differential Geometry III: Exercises

University of Regensburg, Winter term 2021/22
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Please hand in the exercises until **Friday, December 3**

Exercises Sheet no. 6

1. Exercise (4 points).
Let G be a compact Lie group and M be a compact manifold of dimension $n > 2$. Assume that there is a smooth G-action $\Phi\colon G \times M \to M$ on M. We denote by $\bullet^G$ the G-invariant objects of $\bullet$ i. e., $C^\infty(M)^G = \{f \in C^\infty(M) \mid \forall\gamma \in G\colon f \circ \Phi(\gamma, \bullet) = f\}$ is the space of G-invariant smooth funtions, $\text{s-}\mathcal{R}(M)^G = \{g \in \text{s-}\mathcal{R}(M) \mid \forall \gamma \in G\colon \Phi(\gamma, \bullet)^* g = g\}$ is the space of G-invariant semi-Riemannian metrics, etc. We denote by $\mathcal{S}$ the Einstein-Hilbert functional $g \mapsto \int_M \operatorname{scal}^g \operatorname{dvol}^g$.

a) Let $g_0 \in \text{s-}\mathcal{R}(M)^G$. Show that g is a stationary point of $\mathcal{S}\colon [g_0]^G \to \mathbb{R}$ if and only if $\operatorname{scal}^g = 0$.
Hint: Consider the variation $g_t = g + tfg$ for $f = (1 - \frac{n}{2}) \operatorname{scal}^g$. Don't forget to argue why $g_t \in [g_0]^G$ for small t.

b) Characterize stationary points of $\mathcal{S}\colon [g_0]_1^G \to \mathbb{R}$ for $g_0 \in \text{s-}\mathcal{R}(M)^G$.

c) Show that stationary points of $\mathcal{S}\colon \text{s-}\mathcal{R}(M)^G \to \mathbb{R}$ are G-invariant Ricci-flat metrics.
Hint: You may use without proof that a semi-Riemannian metric induces a non-degenerate inner product $\Gamma(\odot^2 T^*M)^G \times \Gamma(\odot^2 T^*M)^G \to \mathbb{R}$ via $\int_M \langle \bullet, \bullet\rangle \operatorname{dvol}$.

2. Exercise (4 points).
Let g be a semi-Riemannian Einstein metric on a manifold M. Show that for $h = fg$, where $f \in C^\infty(M)$ with compact support and $\int_M f \operatorname{dvol}^g = 0$, the second variation of the Einstein-Hilbert functional reduces to

$$\mathcal{S}''_g(h, h) = \frac{n-2}{2} \int_M f\,((n-1)\Delta f - \operatorname{scal} f) \operatorname{dvol}.$$

3. Exercise (4 points).
Let (M, g) be a semi-Riemannian manifold and $u \in C^\infty(M)$. Set $\tilde{g} = e^{2u} g$.

a) Show that for $X, Y, Z, W \in \Gamma(TM)$

$$\begin{aligned} e^{-2u}\tilde{g}(R^{\tilde{g}}(X,Y)Z, W) = g(R^g(X,Y)Z, W) \\ + U(X,Y,Z,W) - U(X,Y,W,Z) - U(Y,X,Z,W) + U(Y,X,W,Z), \end{aligned}$$

where $U(X,Y,Z,W) = \big(\operatorname{Hess} u(X,Z) - \mathrm{d}u(X)\mathrm{d}u(Z) + \frac{1}{2}\|\mathrm{d}u\|^2 g(X,Z)\big)\, g(Y,W)$.

b) Calculate traces to obtain

$$\begin{aligned} \operatorname{ric}^{\tilde{g}}(Y,Z) = \operatorname{ric}^g(Y,Z) - (n-2)(\operatorname{Hess} u - \mathrm{d}u \otimes \mathrm{d}u)(Y,Z) \\ + (\Delta u) g(Y,Z) - (n-2)\|\mathrm{d}u\|^2 g(Y,Z) \end{aligned}$$

for $Y, Z \in \Gamma(TM)$ and

$$e^{2u} \operatorname{scal}^{\tilde{g}} = \operatorname{scal}^g + 2(n-1)\Delta u - (n-1)(n-2)\|\mathrm{d}u\|^2.$$