Differential Geometry III: Exercises

University of Regensburg, Winter term 2021/22
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Please hand in the exercises until **Friday, November 19**

Exercises Sheet no. 4

1. Exercise (1+1+1+0,5+0,5 points).
For a compact orientable manifold M we consider the Einstein-Hilbert functional (with vanishing cosmological constant)

$$\begin{aligned} \mathcal{S} : \Gamma(\textstyle\bigodot^2_{\text{inv}} T^*M) &\to \mathbb{R} \\ g &\mapsto \int_M \operatorname{scal}^g \operatorname{dvol} g \end{aligned}$$

a) Justify why we have $\mathcal{S}(g) = \mathcal{S}(\phi^* g)$ for any orientation-preserving diffeomorphism $\phi\colon M \to M$.

b) Show that for any $X \in \Gamma(TM)$ we have $\mathcal{S}'_g(\mathcal{L}_X g) = 0$. *Hint: Apply a) to the flow ϕ_t^X of X and derive at $t = 0$.*

c) Use b) in order to prove
$$\delta\Big(\operatorname{ric}^g - \frac{1}{2}\operatorname{scal}^g g\Big) = 0.$$
Hint: Exercise sheet no. 3, Exercise 1

d) Justify the relation $\delta(fg) = -df$ for $f \in C^\infty(M)$.

e) Prove
$$\delta \operatorname{ric}^g = -\frac{1}{2} d \operatorname{scal}^g.$$

2. Exercise (4 points).
Let g be a Riemannian metric on a compact manifold M. We assume that g has vanishing scalar curvature (i. e., $\operatorname{scal}^g \equiv 0$), but nowhere vanishing Ricci tensor $\operatorname{ric}^g \neq 0$.

Show that there is a family of metric g_t, $t \in (-\epsilon, \epsilon)$, $g_0 = g$ that simultaneously satisfies

a) For $t > 0$ we have $\operatorname{scal}^{g_t} > 0$.

b) For $t < 0$ we have $\operatorname{scal}^{g_t} < 0$.

c)
$$\delta^g \left(\frac{d}{dt}\Big|_{t=0} g_t \right) = 0.$$

Hint: Write down a family g_t with $(d/dt)|_{t=0} g_t = -\operatorname{ric}^g$ and apply Exercise 1.

3. Exercise (4 points).
We consider a compact Riemannian manifold (M, g). Recall that $\alpha, \beta \in \Omega^1(M) = \Gamma(T^*M)$ are orthogonal if $\int_M \langle \alpha, \beta \rangle_{T^*M} \operatorname{dvol}^g = 0$. For a subspace $V \subset \Omega^1(M)$ we write $V^\perp$ for its orthogonal complement in this sense.

a) Show that $\omega \in \Omega^1(M)$ is in the kernel of $\delta^g \circ (\delta^g)^* : \Omega^1(M) \to \Omega^1(M)$ if, and only if, $\omega^\sharp$ is a Killing vector field, i. e., it satisfies $\mathcal{L}_{\omega^\sharp} g = 0$.

b) Show that if $\alpha \in \Omega^1(M)$ is in the image of δ^g, then $\alpha \in \big(\ker(\delta^g)^*\big)^\perp$.

The following fact may be used without proof in the following – it can be proven by using the fact that $\delta^g \circ (\delta^g)^*$ is a self-adjoint elliptic differential operator:
The map $\delta^g \circ (\delta^g)^*\big|_{(\ker(\delta^g)^*)^\perp}$ is a bijective linear map $\big(\ker(\delta^g)^*\big)^\perp \to \big(\ker(\delta^g)^*\big)^\perp$.

c) Show that for any $h \in \Gamma(\bigodot^2 T^*M)$ we find a vector field X such that $\delta^g\big(h - \mathcal{L}_X g\big) = 0$. This vector field X is unique up to a Killing vector field.

d) We obtain a direct sum decomposition

$$\Gamma(\bigodot^2 T^*M) = \big\{h \in \Gamma(\bigodot^2 T^*M) \,\big|\, \delta^g h = 0\big\} \oplus \big\{\mathcal{L}_X g \,\big|\, X \in \Gamma(TM)\big\},$$

and this decomposition is orthogonal.