Differential Geometry III: Exercises

University of Regensburg, Winter term 2021/22
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Please hand in the exercises until **Friday, November 12**

Exercises Sheet no. 3

1. Exercise (4 points).
Let (M, g) be a semi-Riemannian manifold. Show the following:

a) $\mathrm{sym} \circ \nabla : \Gamma(T^*M) \to \Gamma(\bigodot^2 T^*M)$ is the formal adjoint of $\delta : \Gamma(\bigodot^2 T^*M) \to \Gamma(T^*M)$. This means that for all $h \in \Gamma(\bigodot^2 T^*M)$ and $\omega \in \Gamma(T^*M)$ with $\mathrm{supp}(h) \cap \mathrm{supp}(\omega)$ compact, we have

$$\int_M \langle \mathrm{sym} \circ \nabla\omega, h\rangle \, \mathrm{dvol}^g = \int_M \langle \omega, \delta h\rangle \, \mathrm{dvol}^g .$$

We thus also write δ^* instead of $\mathrm{sym} \circ \nabla$.

b) For any $\omega \in \Gamma(T^*M)$ we have $\delta^*\omega = \frac{1}{2}\mathcal{L}_{\omega^\sharp} g$, where $\mathcal{L}$ is the Lie derivative.[1]

2. Exercise (4 points).
For a function $u \in C^\infty(M)$, prove that the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g} = e^{2u} g$ satisfies

$$\tilde{\nabla}_X Y = \nabla_X Y + (\partial_X u)Y + (\partial_Y u)X - g(X, Y)\,\mathrm{grad}\, u,$$

where $X, Y \in \mathfrak{X}(M)$, ∇ denotes the Levi-Civita connection of g and grad is the gradient with respect to g.

3. Exercise (4 points).
Let (M, g) be a semi-Riemannian manifold, $\mathcal{V}$ a smooth function on M and $L \in C^\infty(TM)$ be given by

$$L(v) = \frac{1}{2} g(v, v) - \mathcal{V}(\pi(v)).$$

Let $q : I \to M$ be a curve that is a stationary point of the action functional of L.

a) Show that $E = \frac{1}{2} g(\dot{q}, \dot{q}) + \mathcal{V}(q)$ is constant along q.

b) Assume that $E - \mathcal{V}(q(t)) = \frac{1}{2} g(\dot{q}(t), \dot{q}(t)) \neq 0$ for all $t \in I$. Show that q is a pregeodesic[2] for the metric $\tilde{g} = (E - \mathcal{V})g$.

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[1] A previous version of the exercise had a wrong sign here, which we copied from Besse, Einstein manifolds, Lemma 1.60. This is obviously a typo, see also Besse, Einstein manifolds, Proof of Theorem 1.81. Sorry, for having copied the sign without previous checking.

[2] Recall: A curve is a pregeodesic if – by definition – a reparametrization of this curve is a geodesic.