Differential Geometry III: Exercises

University of Regensburg, Winter term 2021/22
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Please hand in the exercises until **Friday, November 5**

Exercises Sheet no. 2

1. Exercise (2+1+0.5+0.5 points).
Let (M, g) be a semi-Riemannian manifold, $p \in M$, and $X_p \in T_pM$, $n = \dim M$. The goal of this exercise is to show that X_p extends locally to a synchronous vector field X, defined below. Let V be an open starshaped neighborhood of 0 in T_pM, such that $\exp_p|_V : V \to U := \exp_p(V)$ is a diffeomorphism. Having chosen a generalized orthonormal basis of (T_pM, g_p), one identifies (T_pM, g_p) with $\mathbb{R}^{m,k}$ and then $x := \big(\exp_p|_V\big)^{-1} : U \to V$ defines normal coordinates around p. Let $\gamma : [0,1] \to M$ denote some curve in U.

a) Derive an ordinary differential equation for the function

$$\begin{aligned} [0,1] &\to \mathbb{R}^{2n} \\ t &\mapsto \Big(x^1(\gamma(t)), \dots, x^n(\gamma(t)), Y^1|_t, \dots, Y^n|_t\Big) \end{aligned}$$

in terms of the metric's coefficients g_{ij} and the Christoffel-Symbol Γ_{ij}^k, that holds, if, and only if γ is a geodesic and if $t \mapsto \sum_{i=1}^n Y^i|_t \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$ is parallel along γ.

b) Explain how solutions of this ODE may be used to construct a *smooth* extension $X \in \Gamma(TU)$ of X_p, such that $t \mapsto X|_{\exp_p(tw)}$ is parallel along the geodesic $[0,1] \ni t \mapsto \exp_p(tw)$ for all $w \in V$.

c) Argue why we have $\nabla X|_p = 0$. (Smooth extensions with this property will be called synchronous in p.)

d) We choose a generalized orthonormal basis $(e_1|_p, \dots, e_n|_p)$ of (T_pM, g_p) and we extend each $e_i|_p$ to a synchronous vector field $e_i \in \Gamma(TU)$. Is $(e_1|_q, \dots, e_n|_q)$ a generalized orthonormal frame for each $q \in U$?

2. Exercise (4 points).
Let (M, g) be an oriented semi-Riemannian manifold, with semi-Riemannian volume form dvol^g.

a) For a vector field $X \in \Gamma(TM)$ we define the *divergence* $\operatorname{div} X := \operatorname{tr}_g \langle \nabla_{\bullet} X, \bullet \rangle$. Show that

$$d(X \lrcorner \,\mathrm{dvol}^g) = (\operatorname{div} X)\, \mathrm{dvol}^g\,.$$

Hint: It is advisable to write locally $\mathrm{dvol}^g = e_1^* \wedge \ldots \wedge e_n^*$ *where* $(e_1, \dots, e_n)$ *is a locally defined generalized orthonormal basis of* T_pM*, smoothly depending on* p *and where* $(e_1^*, \dots, e_n^*)$ *is the associated algebraically dual basis of* T_p^*M*. You may assume that the basis is synchronous in* p*, but you also may solve the problem without this additional assumption, which implies additional terms.*

b) $\delta(\omega) = -\operatorname{div}(\omega^\sharp)$, where $\delta : \Omega^1(M) \to C^\infty(M)$ is the divergence defined in Definition 1.2.7 of the lecture and where $\sharp$ is the inverse of $\flat : TM \to T^*M$, $X \mapsto g(X, \bullet)$.

3. Exercise (4 points).
Let (M, g) be a semi-Riemannian manifold. Show that $d : \mathcal{C}^\infty(M) \to \Gamma(T^*M)$ is the formal adjoint of $\delta : \Gamma(T^*M) \to \mathcal{C}^\infty(M)$. This means that for all $f \in \mathcal{C}^\infty(M)$ and $\omega \in \Gamma(T^*M)$ with $\mathrm{supp}(f) \cap \mathrm{supp}(\omega)$ compact, we have

$$\int_M \langle df, \omega \rangle \,\mathrm{dvol}^g = \int_M f \delta\omega \,\mathrm{dvol}^g \,.$$

4. Exercise (4 points).
Let (M, g) be a semi-Riemannian manifold and consider the Lagrange function

$$L(v) = \frac{1}{4} g(v, v)^2 - \frac{1}{2} g(v, v).$$

For $x_1, x_2 \in M$, determine all stationary points $q \in \mathcal{D}_{x_1,x_2}$ of the associated action functional.
Hint: Show first that $g(\dot{q}, \dot{q})$ is constant for the curves $q \in \mathcal{D}_{x_1,x_2}$ that are stationary for the action functional on $\mathcal{D}_{x_1,x_2}$.