Differential Geometry III: Exercises

University of Regensburg, Winter term 2021/22
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Please hand in the exercises until **Monday, October 25**

Exercises Sheet no. 1

1. Exercise (4 points).
Let M be a manifold. Show that the space of connections on M is an affine space over the vector space $\Gamma(T^*M \otimes T^*M \otimes TM)$, i. e. for a connection ∇ and some $A \in \Gamma(T^*M \otimes T^*M \otimes TM)$ a connection is defined by $(X,Y) \mapsto \nabla_X Y + A(X,Y)$ and the difference of two connections ∇^1, ∇^2 provides an element $\nabla^1 - \nabla^2 \in \Gamma(T^*M \otimes T^*M \otimes TM)$. What can be said about torsionfree connections, and – given a metric g on M – about metric connections?

2. Exercise (4 points).
For a semi-Riemannian manifold (M,g), we define the *energy functional* by

$$E(\gamma) = \frac{1}{2}\int_a^b g(\dot\gamma(t),\dot\gamma(t))\mathrm{d}t$$

for a smooth curve $\gamma\colon [a,b] \to M$. Assume that g is Riemannian and recall that the length of a curve γ as above is given by

$$L(\gamma) = \int_a^b \|\dot\gamma(t)\|\mathrm{d}t.$$

a) Show that $L(\gamma)^2 \leq 2E(\gamma)(b-a)$.

b) Now consider the variational problem with fixed end points. Prove that a curve γ is a stationary point of E if and only if it is a stationary point of L and parametrised proportionally to arc-length.

3. Exercise (4 points).
Let (M,g) be a Lorentzian manifold and $a < b$. For non-spacelike curves $\gamma\colon [a,b] \to M$, the proper time was defined by

$$L(\gamma) = \int_a^b \sqrt{-g(\dot\gamma(t),\dot\gamma(t))}\mathrm{d}t.$$

a) Show that $L(\gamma)^2 \leq -2E(\gamma)(b-a)$.

b) Again consider the variational problem with fixed end points. Prove that a timelike curve γ is a stationary point of E if and only if it is a stationary point of L and parametrised proportionally to proper time.