Seminar on the h-principle

Summer term 2021

Prof. Bernd Ammann

Tuesday 16-18, Zoom

Number of sessions: **13**

Available Dates: 13.4., 20.4., 27.4., 4.5., 11.5., 18.5., 25.5., 8.6. (Bernd in Oberwolfach), 15.6., 22.6., 29.6. (Bernd in Oberwolfach), 6.7., 13.7.

Special obstruction:

- May 25: Tuesday after Pentecost – no teaching: we decided to meet anyhow
- June 1: SFB evaluation
- Bernd in Oberwolfach: June 8 and June 29

In the seminar we mainly follow the references [3] and [4]. The advantage of [4] is its concentration on the major ideas, however [3] provides more details and variations. A classical reference, which played a crucial role for the development of the are is [5]. In these textbook style sources there are also many references to original literature.

1 Holonomic Approximation

Talk no. 1: Jets and Thom transversality. *13.4.* Roman Schiessl.
[3, Chap. 1 and 2] In this talk we summarize some preliminaires. The main reference consists of two short chapters [3, Chap. 1 and 2]. In case the audience whishes to have the introductory talks in more details, it is possible to this talk in two parts. Part I: Definition and properties of jets, holonomic sections, geometric representations and holonomic splitting. Part II: Thom transversality theorem. Additional literature [4, Chap. 2] and [13, 14].

Talk no. 2: Open $\mathrm{Diff}(V)$-invariant differential relations. *20.4.* Guadalupe Castillo-Solano.
Explain the notion of an open and invariant differential relation $\mathcal{R}$. Then state and explain (without proofs) Gromov's theorem about the weak equivalence

$$j^r : \Gamma_0 E \xrightarrow{\cong} \Gamma\mathcal{R}, \tag{1}$$

see [4, Theorem 3.3], which we will call the parametric h-principle in the following. The main references are [3, Chap. 7] and [4, Chap. 3], but also explain the notions introduced in [3, Sec. 6.1]. If time admits, then also explain the Parametric Holonomic Approximation [3, Theorem 3.1.2] and how this can be used to prove the application in [3, Section 4.1].

Talk no. 3: Smale immersion theorem and applications to closed manifolds. *27.4. +4.5.* Jonathan Glöckle.

In this talk we consider as differential relation the immersion relation $\mathcal{R}_{\mathrm{imm}} \subset J^1(M,N)$, see one of the exammples in [3, Sec. 5.1]. We mainly follow the presentation of [4, Chap. 3.3.1], however enriched by further literature as, e. g., [3]. In this case the above parametric h-principle, i. e., the weak equivalence (1) says that the differential

$$d : \mathrm{Imm}(M,N) \longrightarrow \mathrm{Mon}(TM,TN) \tag{2}$$

is a homotopy equivalence, provided that M is a connected and non-compact manifold which is [4, Theorem 1.1] in the open case. Then we consider the case M closed with $\dim M < \dim N$. Show in this case [4, Theorem 1.1] or its strengthened form with C^0-denseness [3, Hirsch's Theoerem 8.2.1], using the microextension trick [3, Sec. 8.1] or [4, Sec. 3.3.1]. It is the speaker's choice whether (s)he will treat the special case $M = S^n$, $N = \mathbb{R}^{n+1}$ first, and then the general case later (as in [4]) or whether (s)he will start the general case.

What we should have seen at the end of the talk are proofs of $\pi_0(\mathrm{Imm}(S^1,\mathbb{R}^2)) \cong \mathbb{Z}$ and more generally $\pi_0(\mathrm{Imm}(S^n,\mathbb{R}^{n+1})) \cong \pi_n(\mathrm{SO}(n+1))$.

Supplementary Talk no. 1: Regular homotopy classes of immersions $M^2 \to \mathbb{R}^3$ and (s)pin structures *4.5.* Julian Seipel.

The goal of this talk is to deepen the results of the last talk for surfaces. This talk is probably a very challenging talk as it does not follow the literature in some linear way, and thus it probably requires that the speaker has to have either worked on the subject or invest considerable time. It also leaves a lot of freedom for the speaker to make choices. Some aspects, e. g., the non-orientable case might even lead to some publishable work when worked out in detail.

Note that the elements $\pi_0(\mathrm{Imm}(M,N))$ are called regular homotopy classes. Let M be a connected surface. The goals of talk might be – if achievable –:

(1) to derive an isomorphism from $\pi_0(\mathrm{Imm}(M,\mathbb{R}^3))$ to the space of spin structures on M, provided that M is oriented. [15].

(2) to show (for oriented surfaces:) $\pi_1(\mathrm{Imm}(M,\mathbb{R}^3)) \cong \mathbb{Z}$.

(3) to discuss the non-oriented case using pin structures. This might need some investigations as some publication is different ways of formalizing, see [12], [10].

(4) to discuss the quadratic map $H_1(M,\mathbb{Z}_2) \to \mathbb{Z}_2$ which classifies the regular homotopy class $\in \pi_0(\mathrm{Imm}(M,N))$ of an immersion for oriented surfaces, and possible extension to the non-oriented generalization q_f as described in [12].

(5) Check how this is related to the Arf invariant and the quadratic refinement discussed in [2, Sec. 7.1]

Lecture note's which roughly fit to our needs were written by Tahl Nowik `https://u.cs.biu.ac.il/~tahl/notes.pdf` (extended info) which express similar effects "in another language". One might also consult related work by Nowik as e. g., [10].

Talk no. 4: Smale's sphere eversion. *11.5.* Julian Seipel.
(This talk might be short, maybe just 45 minutes). It follows from Talk no. 4 (and possibly Supplementary Talk no. 2) that the space $\mathrm{Imm}(S^2, \mathbb{R}^3)$ is connected. Thus there are sphere eversion. This allows us to make a relaxing talk with movies and images, but nevertheless based on solid mathematics. There are several movies and some of them are copies of each other, so the speaker should make a choice e. g., among:

- The mpeg video from `http://www.geom.uiuc.edu/docs/outreach/oi/`
- `https://www.youtube.com/watch?v=wO61D9x6lNY`
- `https://www.youtube.com/watch?v=iynrV-3I9CY`

Make a choice among the videos above and maybe some other ones which are available in the internet. Argue why there is no sphere eversion that is invariant under an S^1-action by rotation. Also argue, why the space of sphere eversions is not connected – it has countably many connected components; this is essentially the statement $\pi_1(\mathrm{Imm}(S^2, \mathbb{R}^3)) \cong \mathbb{Z}$.

Talk no. 5: The Phillips submersion theorem. *18.5.* Georgios Raptis.
(This talk might be short, maybe just 45 minutes). In this talk we consider as differential relation the submersion relation $\mathcal{R}_{\mathrm{sub}} \subset J^1(M, N)$, see e. g., [4, Sec. 3.3.2][1].
An additional helpful reference might be the original publication by Phillips [11].

Supplementary Talk no. 2: Applications of the Phillips submersion theorem to TQFT *25.5.* Georgios Raptis.
The aim of this talk is to explain the relevance of the Phillips submersion theorem to the proof of the Galatius-Madsen-Tillmann-Weiss theorem. One might follow Lecture 22 of Dan Freed's lecture notes Bordism: Old and New.

Talk no. 6: Applications to curvature. *8.6.* **(Bernd in Oberwolfach)** Dennis zumbil.
In this talk we apply Gromov's parametric h-principle (1) to metrics with sectional curvature in a given interval (a, b), $a < b$. The goal is to prove that on any connected, non-compact manifold there is a Riemannian metric with sectional curvature in (a, b). We recommend to follow [16].
If time admits, the speaker may check that further conclusions can be achieved by the same method, e. g., that the space of all such metrics is contractible.

[1] Here note the typo that the winding number of the example $S^1 \times \mathbb{R} \to \mathbb{R}$ following the figure ∞ is not 1 as claimed, but 0.

In order to clarify the historic origins let us add that in the special case of negative resp. positive sectional curvature (i. e., $b = 0$ resp. $a = 0$) this result was already mentioned in the PhD thesis of Gromov [6], and related questions are addressed in [5]. The whole result is a straightforward application of the parametric h-principle, so one should expect that experts were aware of this consequence. However[2], no one published this beautiful geometric application before Streil.

Talk no. 7: Proof of the holonomic approximation. *15.6.* N.N..
Prove Gromov's h-principle (1), i. e., Theorem 3.3 in [4]. We recommend to follow Subsection 3.4 and 3.5 in [4]. An alternative source is [3, Chap. 3] in which also different variants of the h-principle are dicussed and proved.

2 Further plan for the seminar

After some weeks we will decide on how to continue at this points. Possible subjects are

- Convex integration.[3] We might follow Part 4 (i.e. Chap. 17 to 21) in [3] or [4, Chap. 4]. Applications can be deduced, among many others
  - Metrics with negative Ricci curvature on closed manifolds of dimension ≥ 3, [9]
  - Nash-Kuiper embedding theorem [7, 8]
  - Theillière, Mélanie. Convex Integration Theory without Integration. ArXiv:1909.04908.
- Local flexibility, see C. Bär and B. Hanke [ArXiv:1809.05703 (pdf), [1]]
- A recent overview article by Gromov Geometric, Algebraic, and Analytic Descendants of Nash Isometric Embedding Theorems. Bull. AMS **54**, Nr. 2, 173–245. More from [5].
- Applications to symplectic geometry. We might follow e. g., Part 3 (i.e. Chap. 9 to 16) in [3].
- Applications to contact geometry. We might follow [4].

Seminar-Homepage

http://www.mathematik.uni-regensburg.de/ammann/hprinciple

[2] to our knowledge. Please provide a reference if you know one
[3] This is currently my prefered choice

Literatur

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