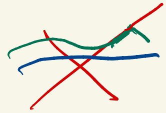
**Recall:** $M$ a time-oriented Lorentzian mfd.

$M$ is globally hyperbolic $\Longleftrightarrow$ $M$ has a Cauchy hypersurface



$\Longleftrightarrow$ $M$ has a Cauchy hypersurface that is a space-like smooth hypersurface

$\Longrightarrow$ $M$ diffeo $S \times \mathbb{R}$ where $S$ is a Cauchy hypers.

**Lemma 2.124**

$M$ globally hyperbolic, $K \subset M$ compact, then $J_\pm(K)$ is closed.

**Prop 2.17** For $A \subset M$: $I_\pm(A) = \mathring{J}_\pm(A)$

**Cor 2.17?** If $B$ is a future set ($I_+(B) \subset B$), then $\partial B$ is an achronal and closed topological hypersurface.

**Note:** For $A \subset M$ $J_+(A)$ is a future set.

**Def.** 2.118 A closed and achronal set $A \subset M$ is *future-trapped* if $J_+(A) \setminus I_+(A)$ is compact.

**Prop** 2.123 Let $M$ be connected, time-oriented, lightlike future-complete Lorentzian mfd with $\mathrm{ric}(X,X) \geq 0$ $\forall X \in TM$ lightlike.

Let $P \subset M$ be a compact, achronal and spacelike (smooth) submfd of codim 2. Assume that the mean curvature vector field $\vec{H}$ of $P$ in $M$ is timelike past-directed.

Then $P$ is future trapped.

__Theorem 2.125__ (Penrose singularity theorem) (Nobel Prize 2020 Physics)

Let $M$ be a connected time-oriented Lorentzian manifold with $ric(\gamma,\gamma) \geq 0 \ \forall \gamma \in TM$ lightlike. Let $S$ be a non-compact a (~~smooth spacelike hypersurface and a~~) Cauchy hypersurface and let $P$ be a non-empty, compact, spacelike and achronal submfd of codim 2 whose mean curvature v.f. is timelike and past-directed. Then $M$ is <u>not</u> lightlike future complete.

Proof: We assume ~~that M~~ is lightlike future complete. $m = \dim M = n+1$

a) M has a Cauchy hypersurface
$\Rightarrow$ M is glob. hyp.

$\overset{L.2.1}{\Rightarrow}$ $J_+(P)$ is closed

$$\underbrace{J_+(P)}_{\text{closed}} \setminus I_+(P) \overset{P2.17}{=} \overline{J_+(P)} \setminus \mathring{J}_+(P)$$

$$= \partial J_+(P)$$

By Cor ~~2.8~~, $J_+(P)$ future set

$\partial J_+(P)$ is an achronal topological hypersurface.

Prop 2.123 $\Rightarrow$ $\partial J_+(P)$ is compact.

b) <u>Claim</u> $\partial \mathcal{Y}_\epsilon(P) \neq \emptyset$.

Suppose $\partial \mathcal{Y}_\epsilon(P) = \emptyset$. Then $\mathcal{Y}_\epsilon(P) = \underline{I}_\epsilon(P)$
closed open

$\Downarrow$

$P \subset$

$\mathcal{Y}_\epsilon(P) = \emptyset$ or $\mathcal{Y}_\epsilon(P) = M$

$\emptyset \neq P \subset \mathcal{Y}_\epsilon(P)$ ↯

$\mathcal{Y}_\epsilon(P) = \underline{I}_\epsilon(P)$

$\underline{I}_\epsilon(P) \cap P = P \neq \emptyset$

not achronal

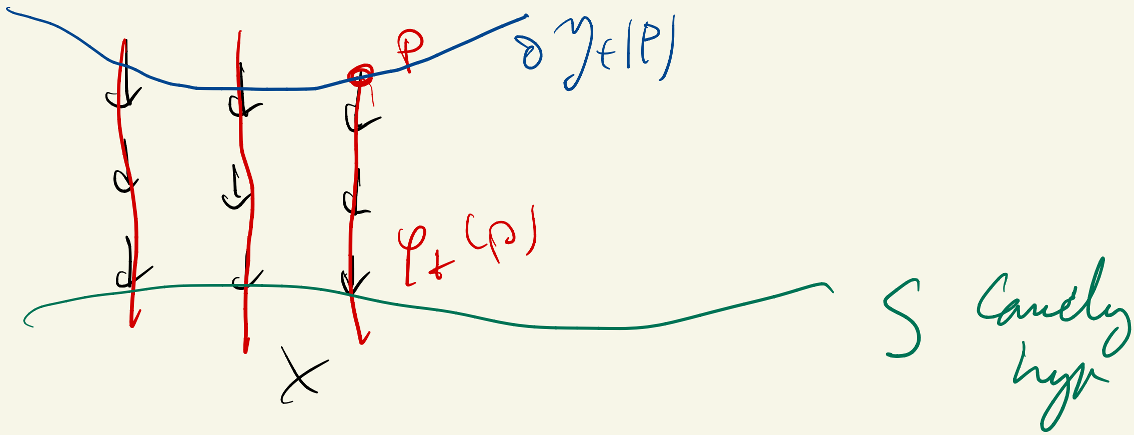
↯

$\Rightarrow$ Claim

c) Choose a timelike past-directed vector vector field $X$ on $M$.

For $p \in M$ we define $\varphi_t(p)$ as the solution of

$$\frac{d}{dt}\varphi_t(p) = X|_{\varphi_t(p)}, \quad \varphi_0(p) = p$$




$t \mapsto \varphi_t(p)$ is a timelike past-directed past and future inextendible.

$\gamma_t$ thus intersects $S$ in precisely one

$t = t_p$, i.e. $\varphi_{t_p}(p) \in S$.

[Similar to previous arguments we see that $p \mapsto t_p$ is continuous]

Define $g : \partial J_\epsilon(p) \longrightarrow S$, $p \mapsto \varphi_{t_p}(p)$

continuous. If $g(p_1) = g(p_2)$,

then $p_1$ and $p_2$ are on the same flow line of $X$, $p_i \in \partial J_\epsilon(p)$. As $\partial J_\epsilon(p)$ is achronal, we have $p_1 = p_2$.

$\Rightarrow g$ injective

$g$ is an injective continuous map from an $n$-dim top mfd to an $n$-dim top. mfd. (*)

# Brower's thm:

If (*) holds then $g(U)$ is open
for any open $U$.
(in other words: $g: \partial(\mathcal{J}_\epsilon(P)) \to Q$
$Q := g(\partial \mathcal{J}_\epsilon(P))$ is a homeom
and $Q \subset S$ is open).

---

$Q$ is open ; $\partial \mathcal{J}_\epsilon(P)$ cpct
$\Rightarrow$ $Q$ cpct. $\Rightarrow$ $Q$ closed

$M$ connected $\Rightarrow$ $S$ connected
$\cong$
$S \times \mathbb{R}$ $\Rightarrow$ ~~$\mathcal{J}$~~ $Q = \emptyset$ or $Q = S$
cpct non-cpct
$\partial \mathcal{J}_\epsilon(P) = \emptyset$ ↯ b) ↯ $\square$

Example (with Exercise 4 on sheet 11) and Exercise 1 on sheet 13

Exterior Schwarzschild solution

$m = \dim M = n+1$

$m > 0 \qquad h(r) = 1 - \frac{2m}{r^{n-2}}$

$M := \mathbb{R} \times \left((2m)^{\frac{1}{n-2}}, \infty\right) \times S^{n-1}$

$g = -h(r)\, dt \otimes dt + \frac{1}{h(r)}\, dr \otimes dr + r^2 g_{S^{n-1}}$

$S := \{0\} \times \left((2m)^{\frac{1}{n-2}}, \infty\right) \times S^{n-1}$

Cauchy hypersurface in $M$.

with unit normal $\frac{1}{\sqrt{h(r)}}\frac{\partial}{\partial t}$,

totally geodesic.

$$P := 0 \times \{r_0\} \times S^{n-1}$$
$$\subset S$$

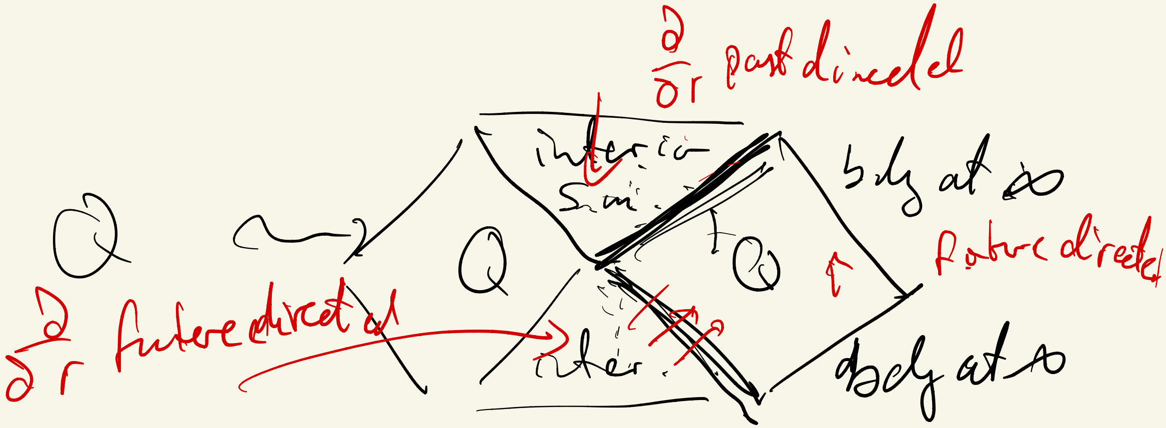
mean curv. v.f. of $P$ in $S$

= mean curv. v.f of $P$ in $M$

There it is spacelike.

$\Rightarrow$ Theorem 2.125 can not be applied

$Q$ = Exterior solution




Interior Schwarzschild solution

As in the exterior solution, but replace $\left((2m)^{\frac{1}{n-2}}, \infty\right)$ by $\left(0, (2m)^{\frac{1}{n-2}}\right)$. $\frac{\partial}{\partial r}$ is timelike, $\frac{\partial}{\partial t}$ spacelike. Assume $\frac{\partial}{\partial r}$ is past-directed.

$S := \mathbb{R} \times \{r_0\} \times S^{n-1}$ Cauchy hyp. s. of the int. Schw.-solution.

non-cpt

$T := \{0\} \times \left(0, (2m)^{\frac{1}{n-2}}\right) \times S^{n-1}$

totally geodesic

$P := \{0\} \times \{r_0\} \times S^{n-1} = S \cap T$.

mean curv. $\vec{H}$ of $P$ in $T$

= mean curv. of $P$ in the interior Schw.-sol.

$\Rightarrow$ it is timelike ; $\vec{II}$ the second f. of $P$ in $T$ or Schwar-sol.

$$\langle \vec{II}(X,Y), \frac{\partial}{\partial r} \rangle|_{r=r_0} = -\underbrace{\frac{1}{2}\partial_r(r^2)}_{r_0} g_{S^{n-1}}(X,Y)$$

$$\langle \vec{H}, \frac{\partial}{\partial r} \rangle|_{r_0} = -\frac{1}{r_0} \cdot 1$$

$$\vec{H} = c(r_0)\frac{\partial}{\partial r}$$

$$c(r_0)\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle = -\frac{1}{r_0}$$

$$= c(r_0)\frac{1}{h(r)} \Rightarrow c(r_0) = \frac{|h(r_0)|}{r_0}$$

$$c(r_0) = \frac{1}{r_0} \underbrace{\left( \frac{2m}{r_0^{n-2}} - 1 \right)}_{-h(r_0) = |h(r_0)|} > 0$$

$\vec{H}$ past-directed timelike.

Therefore Penrose sing. theorem may be applied for $0 < r_0 < (2m)^{\frac{1}{n-2}}$

$\Rightarrow \exists$ lightlike geodesics which can be extended up to $\infty$.

(photons falling in the black hole).

# Solution of Ex 2 of Sheet 13

a) $N \subset M$

spacelike hypersurf. Lorentz. $+$


$\nu$ future directed timelike unit normal v.f.

$$\Phi(t,p) := \exp_p(t\nu|_p)$$

p fixed

$\gamma : t \longmapsto \Phi(t,p)$ timelike geodesic

$\dot\gamma = \nu|_p \perp N$

$\langle \dot\gamma, \dot\gamma \rangle = -1$

$$\dot\gamma(t) = \frac{d}{dt}\Phi(t,p) = d\Phi\left(\frac{\partial}{\partial t}\right)$$

$$\langle d\Phi\left(\frac{\partial}{\partial t}\right), d\Phi\left(\frac{\partial}{\partial t}\right)\rangle = \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle$$

$$= -1$$

$$\parallel$$

$$\Phi^* g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)$$

Now let $X \in T_p N$, $c : (-\varepsilon, \varepsilon) \to N$,

$\dot{c}(0) = X$, $c(0) = p$.

$\gamma_s(t) = \Phi(t, c(s))$ geodesic variation of $\gamma$.

$$d\Phi(X)\Big|_{t,p} = \frac{\partial}{\partial s}\Big|_{s=0} \Phi(t, c(s)) =: J(t)$$

Jacobi v.f.

$$\Phi^* g = -dt \otimes dt + \dots$$

$$y(0) = X \perp \dot{\gamma}(0) \quad \text{as } \Phi(0,p) = p .$$

$$\frac{\nabla}{dt} y(0) = \frac{\nabla}{dt}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} \Phi(t, c(s))$$

$$= \frac{\nabla}{ds}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} \Phi(t, c(s)) = \frac{\nabla}{ds}\Big|_{s=0} \upsilon\big|_{c(s)}$$

$$= - S_{\upsilon}(\underbrace{\dot{c}(0)}_{=X}) \in T_p N$$

$$\perp \dot{\gamma}(0)$$

$$\Rightarrow \quad y(t) \perp \dot{\gamma}(t) \quad \forall t$$

$$\Rightarrow \langle \underbrace{d\Phi(X}_{y(t)}{}_{t,p}) , \underbrace{d\Phi(\tfrac{\partial}{\partial t}|_{t,p})}_{\dot{\gamma}(t)} \rangle = 0$$

Define $h_t := (\underbrace{\Phi(t,\cdot)}_{N \to M})^* g$

$h_t$ is a Riem. metric as

long as $\Phi(t,\cdot)$ is an immersion

in part. for $t$ close to $0$.

$\Rightarrow \Phi^* g = -df \otimes df + h_t$

b) directly follows from Prop 2.52 a)

in the part. notes

c) $\hat{g} = dt \otimes dt + h_t$

~~For~~ Var. formula for $\mathcal{L}^{\varepsilon} g$

is ... :

Exercise groups.