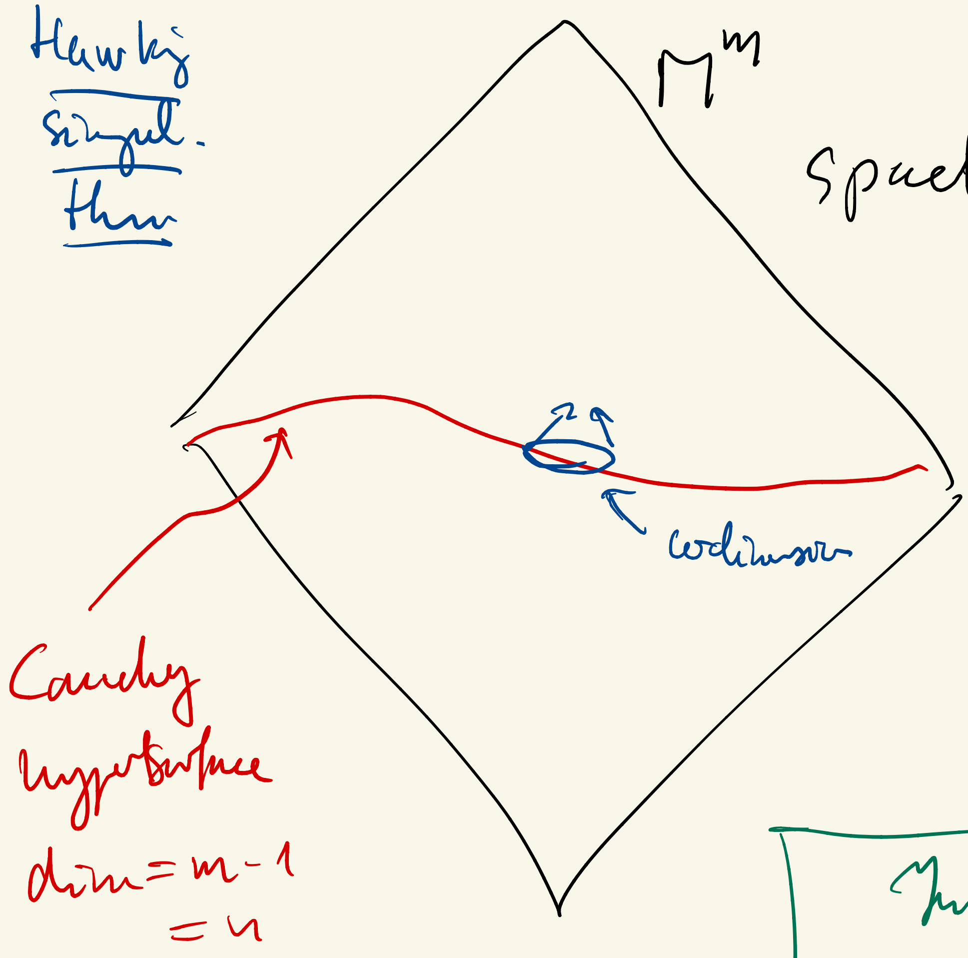
Hawking singul. thm



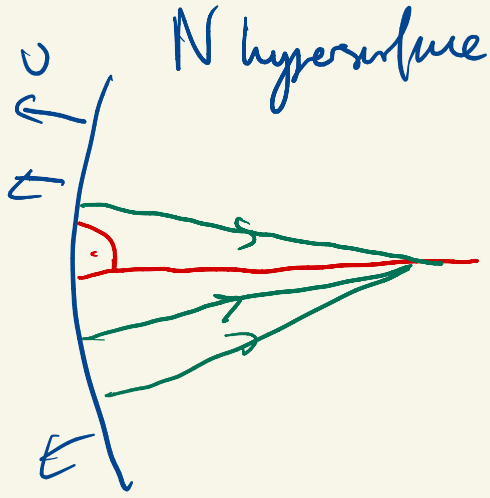
Spacetime

$$\boxed{\begin{array}{l}\mathrm{Ric}(\dot\gamma,\dot\gamma)\geq 0 \\ \text{for all lightl. } \gamma\end{array}}$$

$\Uparrow$ strong energy $\Uparrow$ dominant energy condition

$$\boxed{\begin{array}{l}\text{Inf. version of the geodesic var. starting orthogonally} \\ \frac{\nabla}{dt} J(0) = - S_{\dot c(0)}(J(0))\end{array}}$$

Recall $\mathbb{R}^m$ eucl.



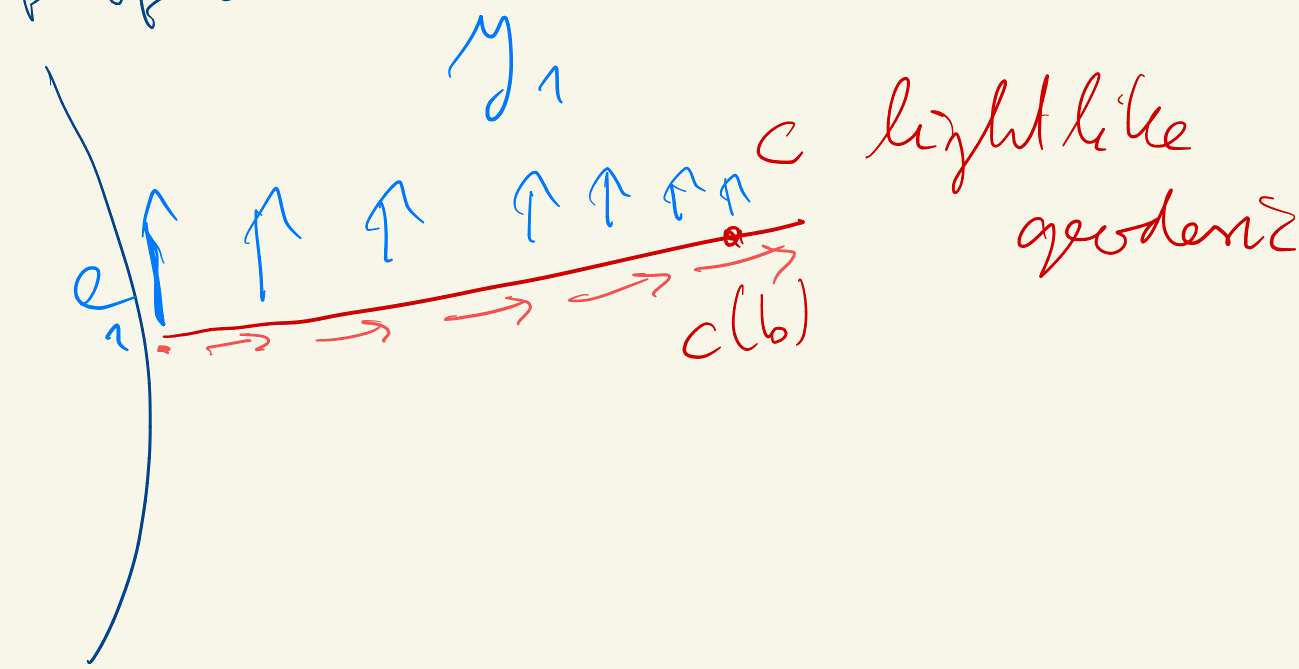
straight line in $\mathbb{R}^m$ = geodesic

ν unit normal $\rightsquigarrow$ II $\rightsquigarrow$ H

If $H \geq \frac{1}{b}$ $\quad b>0$

There is a focal point along c in $[0,b]$

P of codim 2



Prop 2.122 Let M be a Lorentzian mfd, $P \subset M$ a spacelike submfd of codim 2 and mean curvature vector field $\vec{H}$.

Let $c : [0,b] \to M$ be lightlike geodesic, $\underset{p}{c(0)} \in P$, $\dot{c}(0) \perp T_{c(0)}P$

Assume (i) $\mathrm{ric}(\dot{c}(t), \dot{c}(t)) \geq 0 \quad \forall t \in [0,b]$

(ii) $\langle \vec{H}(p), \dot{c}(0) \rangle \geq \frac{1}{b}$

Then c has a focal point in $(0,b]$.

$p \in P$, $(T_pP)^{\perp} = N_pP \cong \mathbb{R}^{1,1}$

Proof: Assume that c has no focal point in $(0,b]$.

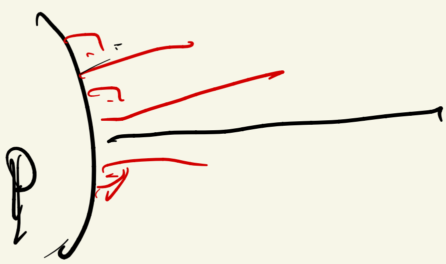
a) Let $e_1, \dots, e_{m-2}$ be an onb of T_pP. $J_i \in \Gamma(c^*TM)$ Jacobi field with

$i \in \{1, \dots, m-2\}$ $\quad J_i(0) = e_i$,

$$\frac{\nabla J_i}{dt}(0) = -S_{\dot c(0)}(e_i)$$

$i = 0 \qquad J_0(0) = 0, \quad \frac{\nabla J_0}{dt}(0) = \dot c(t)$

$$J_0(t) = t \cdot \dot c(t)$$



b) <u>Claim</u> $J_0(t), \dots, J_{m-1}(t)$

is a basis of $(\dot{c}(t))^\perp$

<u>for all</u> $t \in (0, b]$.

$\langle J_0(t), \dot{c}(t)\rangle = t \underbrace{\langle \dot{c}(t), \dot{c}(t)\rangle}_{=0} = 0$

<u>$i > 0$</u>: $\frac{d^2}{dt^2}\langle J_i, \dot{c}\rangle = \langle \frac{D^2}{dt^2} J_i, \dot{c}\rangle$

$= \langle R(\dot{c}, J_i)\dot{c}, \dot{c}\rangle = 0$

$\frac{d}{dt}\langle J_i(t), \dot{c}(t)\rangle \Big|_{t=0} = \langle \frac{D}{dt} J_i, \dot{c}\rangle \Big|_{t=0}$

$= -\langle \underbrace{S_{\dot{c}(0)}(e_i)}_{\in T_pP}, \underbrace{\dot{c}(0)}_{\perp T_pP}\rangle = 0$

$$< y_i(0), \dot{c}(0) > = < e_i, \dot{c}(0) > = 0$$

$$\Rightarrow \quad y_i(t) \perp \dot{c}(t) \quad \forall t .$$

<u>Lin. independence</u> of $y_0(t), \dots, y_{m-2}(t)$

Assume $t_0 \in (0, b]$

$$\sum_{i=0}^{m-2} \alpha_i \, y_i(t_0) = 0 , \quad \alpha_i \in \mathbb{R}$$

not all are 0 .

$$y := \sum_{i=0}^{m-2} \alpha_i \, y_i \quad , \quad y(t_0) = 0$$

$$y(0) = \sum_{i=1}^{m-2} \alpha_i \, e_i \in T_p P$$

$$\pi^{tan}\left(\frac{Dy}{dt}(0)\right)$$

$$= \pi^{tan}\left(\sum_{i=0}^{m-2} \alpha_i \frac{Dy_i}{dt}(0)\right)$$

$$= \pi^{tan}\left(\sum_{i=1}^{m-2} \alpha_i \left(-1\right) S_{\dot{c}(0)}(e_i)\right)$$

$$= -\pi^{tan}\left(S_{\dot{c}(0)}\left(\underbrace{\sum_{i=1}^{m-2} \alpha_i e_i}_{= y(0)}\right)\right)$$

$$= -\pi^{tan}\left(S_{\dot{c}(0)}\left(y(0)\right)\right)$$

$y \not\equiv 0$ $\quad$ t_0 is a focal point $\lightning$

Thus $y_0(t), \dots, y_{m-2}(t)$ are lin. independent $\Rightarrow$ b)

c) <u>Claim</u>

$\forall i,j \in \{0, \dots, m-2\}$

$$\left\langle \frac{D}{dt} y_i , y_j \right\rangle = \left\langle y_i , \frac{D}{dt} y_j \right\rangle$$

$$\frac{d}{dt}\left(\left\langle \frac{D}{dt} y_i , y_j \right\rangle - \left\langle y_i , \frac{D}{dt} y_j \right\rangle \right)$$

$$= \left\langle \frac{D^2}{dt^2} y_i , y_j \right\rangle - \left\langle y_i , \frac{D^2}{dt^2} y_j \right\rangle$$

$$= \left\langle R(\dot c, y_i)\dot c , y_j \right\rangle$$

$$- \left\langle R(\dot c, y_j)\dot c , y_i \right\rangle$$

$= 0$

block symmetry

$j \geq 1, i \geq 1$

$p = c(0) \in P$

$$\left\langle \frac{D y_i}{dt}(0), y_j(0) \right\rangle - \left\langle y_i(0), \frac{D}{dt} y_j(0) \right\rangle$$

$$= \left\langle -S_{\dot{c}(0)}(y_i(0)), y_j(0) \right\rangle$$

$$+ \left\langle y_i(0), S_{\dot{c}(0)}(y_j(0)) \right\rangle$$

$$= 0$$

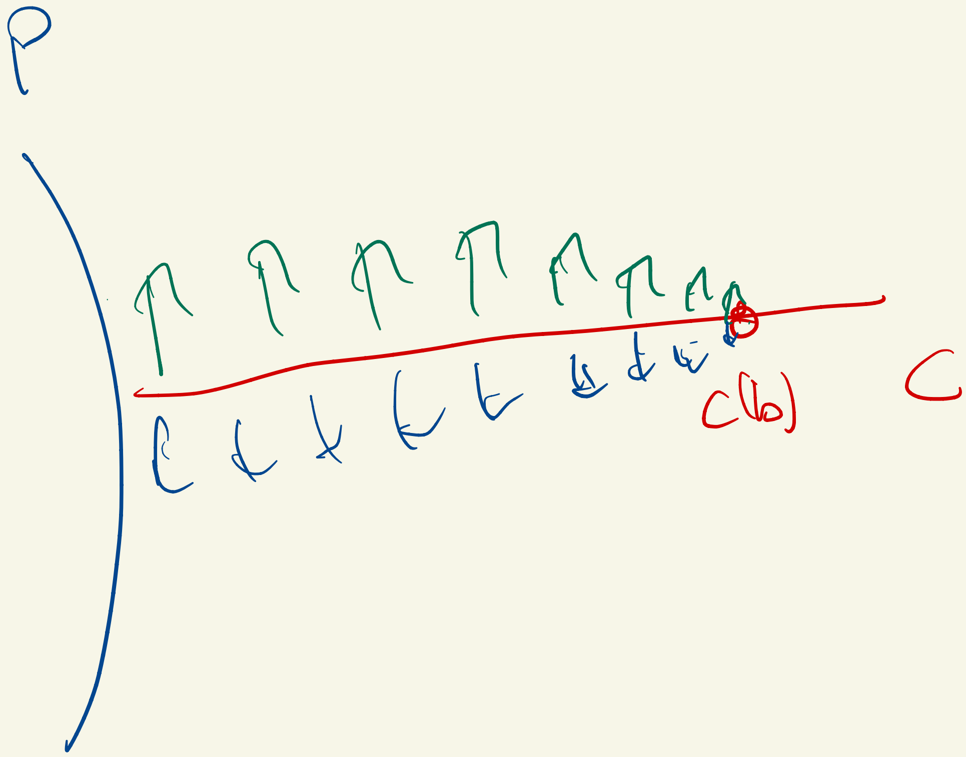
$\Rightarrow$ claim for $i, j \geq 1$

$j \geq 1, i = 0$

$$\left\langle \frac{D y_0}{dt}(0), y_j(0) \right\rangle - \left\langle y_0(0), \frac{D}{dt} y_j(0) \right\rangle$$

$$= \langle \overbrace{\dot{c}(0), e_j}^{=0} \rangle - \langle \underbrace{0, \tfrac{D}{dt} y_j(0)}_{=0} \rangle$$

$\Rightarrow$ claim.


P
C(b)
C

d) For vector fields $V, W \in \Gamma(c^* TM)$ with $W, V \perp \dot{c}$, $W(0), V(0) \in T_p P$ we define the index form

$$I(V,W) = I^{P,c,b}(V,W) = \int_0^b \left[\left\langle \frac{\nabla V}{dt}, \frac{\nabla W}{dt} \right\rangle - \langle R(\dot{c}, W)V, \dot{c} \rangle \right] dt$$

$$- \langle \dot{c}(0), \vec{\mathrm{II}}(V,W) \rangle$$

<u>Claim</u> $I(V,V) \geq 0$ for all V as above with $V(b) = 0$. Equality holds iff $V(t) \parallel \dot{c}(t)$ $\forall t \in [0,b]$.

Note: $I(V,W)$ is the Hessian for the energy $E(\gamma) = \frac{1}{2}\int_0^b \langle \dot\gamma, \dot\gamma \rangle \, dt$

$\gamma : [0,b] \to M$, $\dot\gamma(0) \perp T_p P$, $\gamma(b) = 0$

$\gamma(0) = p$

Any V as in the claim can be written as

$$V(t) = \sum_{i=0}^{m-2} f_i(t) \, y_i(t) \quad \forall t \in [0,b]$$

$$\frac{\nabla V}{dt} = \underbrace{\sum_{i=0}^{m-2} \dot f_i(t) \, y_i(t)}_{X :=} + \underbrace{\sum_{i=0}^{m-2} f_i(t) \frac{\nabla y_i}{dt}(t)}_{Y}$$

$$\left\langle \frac{DV}{dt}, \frac{DV}{dt} \right\rangle = \left\langle X, \frac{DV}{dt} \right\rangle + \left\langle Y, \frac{DV}{dt} \right\rangle$$

$$= \langle X, X \rangle + \langle X, Y \rangle + \frac{d}{dt}\langle Y, V \rangle - \left\langle \frac{D}{dt} Y, V \right\rangle$$

$$= \frac{d}{dt}\langle Y, V \rangle + \langle X, X \rangle$$

$$+ \cancel{\sum_{i,j=0}^{m-2} \dot{f}_i(t) f_j(t) \left\langle y_i(t), \frac{D y_j}{dt}(t) \right\rangle}$$

Claim 4

$$+ \cancel{\sum_{i,j=0}^{m-2} \dot{f}_i(t) f_j(t) \left\langle \frac{D y_i}{dt}(t), y_j(t) \right\rangle}$$

$$+ \sum_{i=0}^{m-2} \left\langle f_i(t) \frac{D^2 y_i}{dt^2}, V \right\rangle$$

$$= \frac{d}{dt} \langle Y, V \rangle + \langle X, X \rangle$$

$$+ \sum_{i=0}^{m-2} \langle f_i(t) R(\dot c(t), Y_i(t)) \dot c(t), V(t) \rangle$$

$$= \frac{d}{dt} \langle Y, V \rangle + \langle X, X \rangle$$

$$+ \langle R(\dot c(t), V(t)) \dot c(t), V(t) \rangle$$

For $\varepsilon > 0$ small we get

$$\int_\varepsilon^b \langle X, X \rangle dt = \int_\varepsilon^b \Big[\langle \frac{\nabla V}{dt}, \frac{\nabla V}{dt} \rangle - \langle R(\dot c, V) \dot c, V \rangle$$

$$- \frac{d}{dt} \langle Y, V \rangle \Big] dt$$

$$= \int_{\varepsilon}^{b} \left[\left\langle \frac{DV}{dt}, \frac{DV}{dt} \right\rangle + \langle R(\dot c, V)V, \dot c \rangle \right] dt$$

$$+ \langle Y(\varepsilon), V(\varepsilon) \rangle \qquad \boxed{V(b) = 0}$$

$$V(t) = f_0(t)\, \dot c(t) + \sum_{i=1}^{m-2} f_i(t) J_i(t)$$

lin. combination of $\dot c(t), J_1(t), \dots, J_{m-2}(t)$

$\Rightarrow$ $F_0(t) = f_0(t)\, t$ has a smooth extension to $[0, b]$

$V(0) \in T_p P$, $F_0(0) = 0$.

$$V(0) = \sum_{i=1}^{m-2} f_i(0) J_i(0) = \sum_{i=1}^{m-2} f_i(0) e_i$$

$$\langle Y(\varepsilon), V(\varepsilon)\rangle$$

$$= \langle V(\varepsilon), \cancel{f_0(\varepsilon)\mathring{c}(\varepsilon)}\rangle \qquad V \perp \mathring{c}$$

$$+ \left\langle V(\varepsilon), \sum_{i=1}^{m-2} f_i(\varepsilon) \frac{D}{dt} y_i(\varepsilon)\right\rangle$$

$$\xrightarrow{\varepsilon \to 0} \sum_{i=1}^{m-2} \langle V(0), f_i(0)\left(-S_{\mathring{c}(0)}(y_i(0))\right)\rangle$$

$$= - \langle V(0), S_{\mathring{c}(0)}(V(0))\rangle$$

$$= - \langle \mathrm{II}(V(0), V(0)), \mathring{c}(0)\rangle$$

$$\Rightarrow I(V,V) = \int_0^b \underbrace{\langle X, X\rangle}_{\geq 0} \, dt \geq 0$$

On $(\mathring{c}(t))^\perp$ the scalar product is pos. semi definit

$\langle X, X \rangle = 0$ iff $X \parallel \mathring{c}$

as if A is a complement of $\mathbb{R}\mathring{c}(t)$ in $(\mathring{c}(t))^{\perp}$ then $\langle\ ,\ \rangle|_{A \times A}$ positiv definit.

$I(V, V) = 0$ iff $X \parallel \mathring{c}$

iff $\mathring{f}_i = 0 \quad \forall i \in \{1, \ldots, m-2\}$

iff $f_i = 0 \quad \forall i \in \{1, \ldots, m-2\}$

$(f_1(b) = \ldots = f_{m-2}(b) = 0)$

iff $V \parallel \mathring{c}$ on $[0, b]$

$V(t) = f_0(t) t \, \mathring{c}(t) \Rightarrow d)$

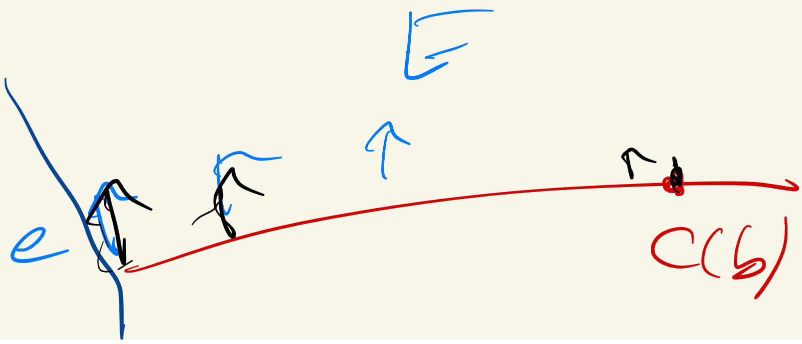
c) For any $e \in T_p P$ let $E_e \in \Gamma(c^* TM)$ with $E_e(a) = e$ and $\frac{\nabla}{dt} E_e = 0$.

Define $V_e(t) := \left(1 - \frac{t}{b}\right) E_e(t)$

$$\frac{\nabla}{dt} V_e = -\frac{1}{b} E_e(t)$$

V_e satisfies conditions in d) and $V_e(b) = 0$.

$$\left(\Rightarrow I(V_e, V_e) \geq 0 \right)$$



V_e is not tangential to $\dot{c}$ for $e \neq 0$

$$\left(\overset{\text{dJ}}{\Longrightarrow} I(V_e, V_e) > 0 \right) \qquad \|e\|=1$$

$$0 < \int_0^b \Big[\underbrace{< \frac{\nabla V_e}{dt}, \frac{\nabla V_e}{dt} >}_{=\frac{1}{b^2}} - < R(\dot{c}, V_e) V_e, \dot{c} > \Big] dt$$

$$- < \dot{c}(0), \vec{\mathrm{II}}(V_e, V_e)\Big|_{t=0} >$$

$$= \frac{1}{b} - \int_0^b \left(1 - \frac{t}{b}\right)^2 < R(\dot{c}, E) E, \dot{c} > dt$$

$$- < \dot{c}(0), \vec{\mathrm{II}}(e, e) >$$

Now let $e_1, \ldots, e_{m-2}$ be an onb of a complement of $\mathbb{R}\dot{c}(t)$ in $(\dot{c}(t))^\perp$, let e run through $e_1, \ldots, e_{m-2}$

and sum over all these e_i :

$$0 < \frac{m-2}{b} - \int_0^b \left(1-\frac{t}{b}\right)^2$$

$$\underbrace{\sum_{i=1}^{m-2} \langle R(\dot c, E_{e_i}) E_{e_i}, \dot c \rangle}_{ric(\dot c(t), \dot c(t)) \geq 0} \, dt$$

$$- (m-2) \underbrace{\langle \dot c(0), \vec{H} \rangle}_{\geq \frac{1}{b}}$$

$$\leq \frac{m-2}{b} - \frac{m-2}{b} \leq 0 \quad \lightning$$

$\square$

Lemma 2.124

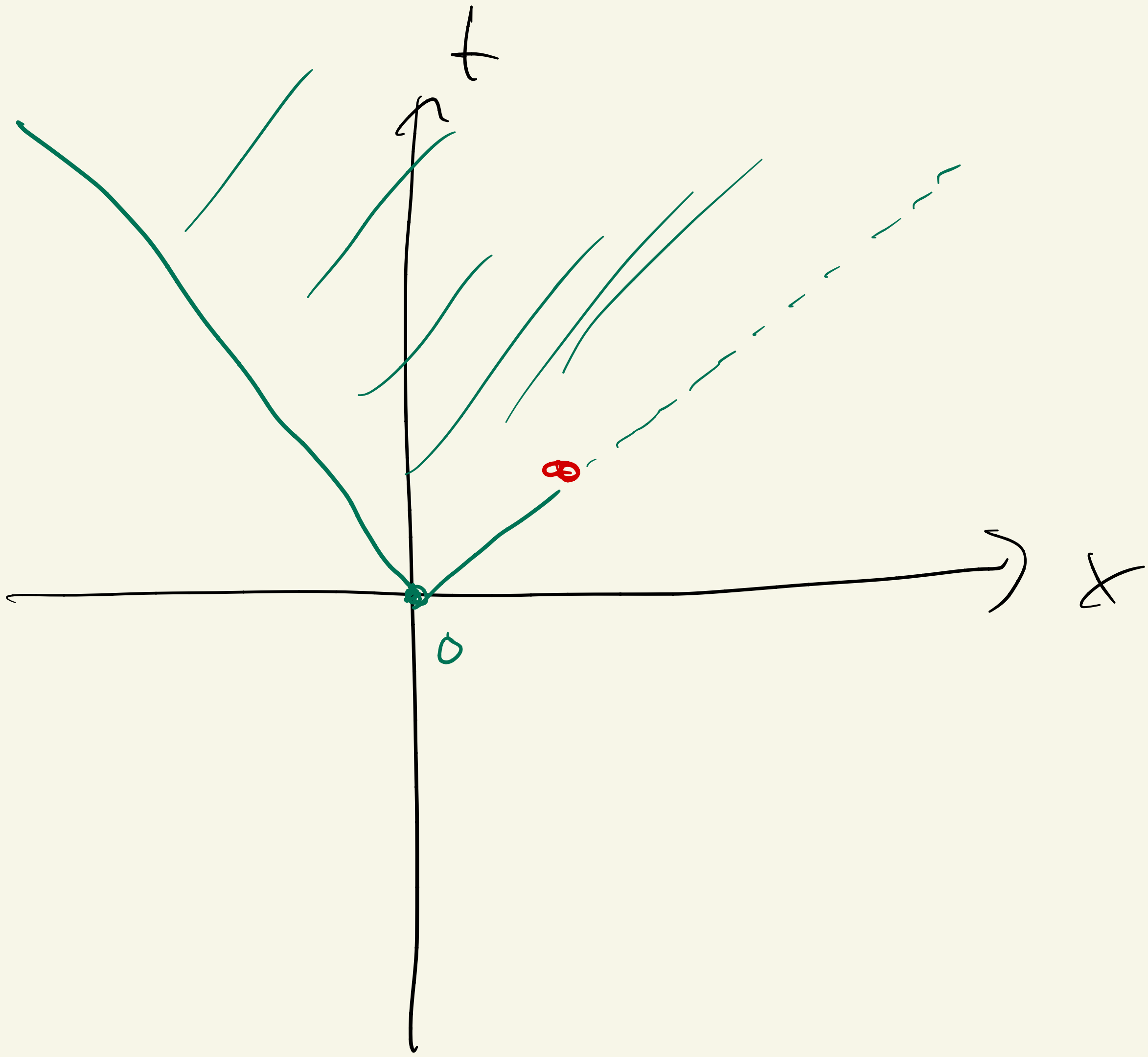
Let M be a glob. hyperbolic Lorentzian manifold let $K \subset M$ be compact. Then $J_\pm(K)$ is closed.

Example "globally hyp." needed

$M = \mathbb{R}^{1,1} \setminus \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, $K = \{0\}$

$$J_+(K) = \left\{ \begin{pmatrix} t \\ x \end{pmatrix} \;\middle|\; \begin{array}{l} t > |x| \text{ or} \\ (t = |x| \text{ and } x < 1) \end{array} \right\}$$

not closed


t
x
0

Use Prop 2.92 (see Bär for proof)

If M is glob. hyp., then $\leq$ is closed relation, i.e. if we have $p_i \to p$, $q_i \to q$, $p_i \leq q_i$, then we also have

$$p \leq q.$$

Proof of Lemma: Let $p_i \in J_+(K)$, $p_i \to p$. Choose $q_i \in K$ with $p_i \geq q_i$. After passing to a subsequence $q := \lim_{i \to \infty} q_i$ exists in K

$\overset{\text{Prop}}{\Longrightarrow}$ $p \geq q \Rightarrow p \in J_+(q) \subset J_+(K)$. □