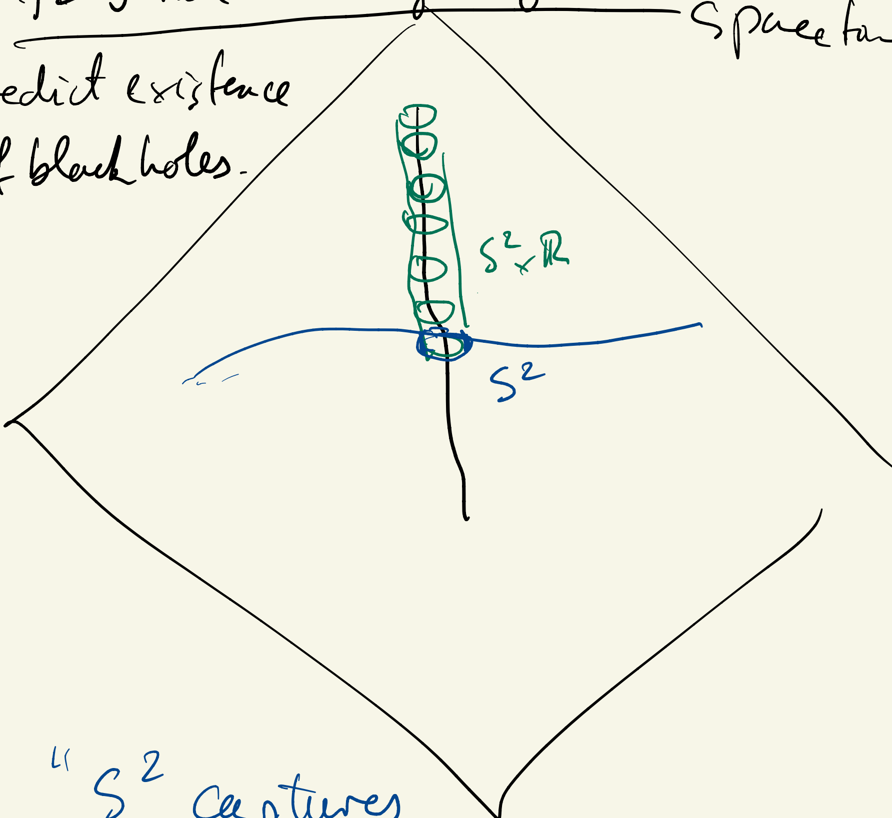
(3.)2.9 Penrose singularity theorem

Predict existence of black holes.

Spacetime (3+1)-dim



"S^2 captures the black hole"

$\rightsquigarrow$ codim 2 spacelike hypersurfaces will be important

Def 2.166

A connected time-oriented Lorentzian mfd M is

timelike future-complete

timelike past-complete

if any future-directed / past-directed timelike geodesic $t \mapsto \exp(tX)$ may be defined on $[0,\infty)$

Similarly: lightlike future-complete / past-complete

Note Lemma 2.51 (Bär)

Let M be a semi-Rie. mfd,

$c : [0,b) \to M$ a geodesic

Then TFAE

(i) $\lim_{t \nearrow b} c(t)$ exists in M

(ii) $\exists \, \varepsilon > 0$ s.th. c can be extended to a geodesic $c : [0, b+\varepsilon) \to M$.

Pf: (ii) $\Rightarrow$ (i), (i) $\Rightarrow$ (ii) Bär

M Lorentzian

Thus M is not timelike future-complete

$\Longleftrightarrow$ $\exists$ f.d. timelike geodesic $\gamma : (a,b) \to M$ such that $\lim_{t \nearrow b} \gamma(t)$ does not exist.

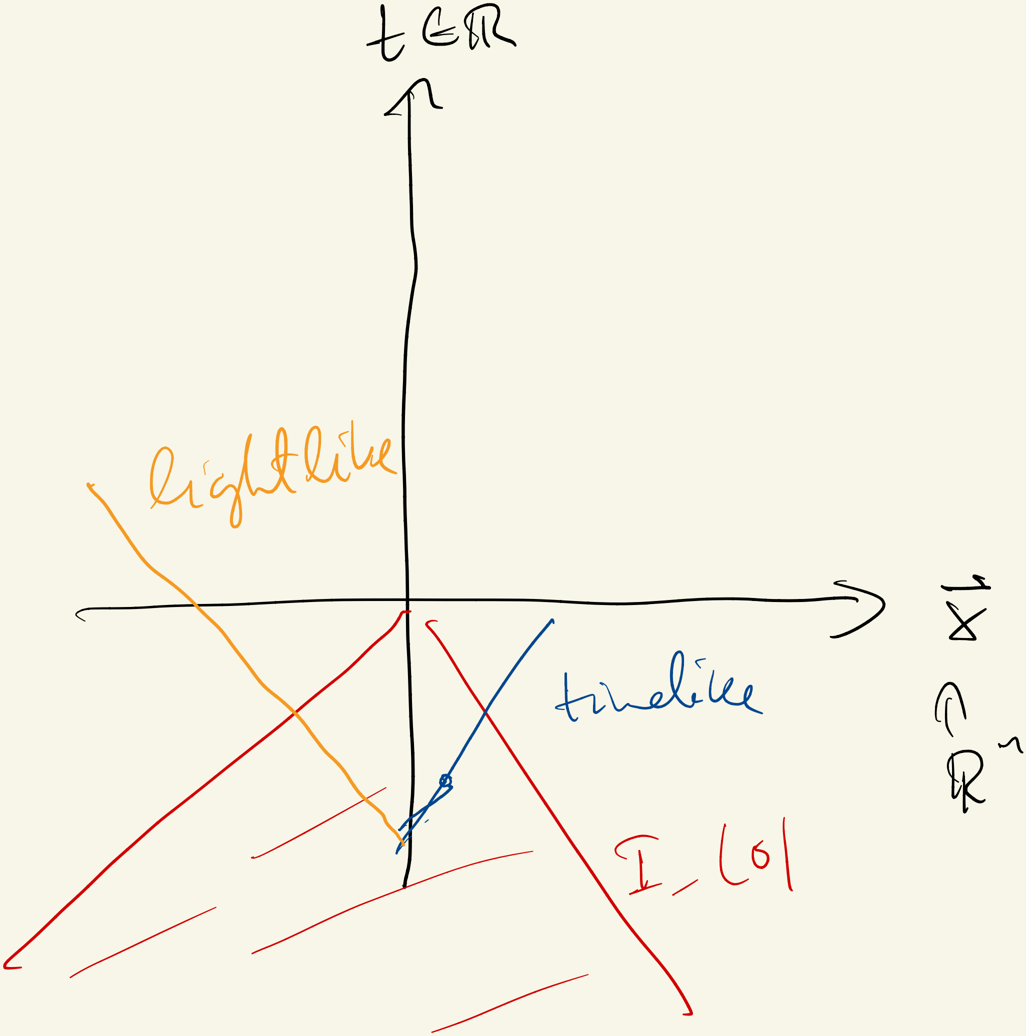
Example

$M = I_-(0) \subset \mathbb{R}^{n,1}$

not timelike future-complete

not lightlike future-comple.

timelike and lightlike past-complete.



$A \subset M$ $\qquad E_\pm(A) := J_\pm(A) \setminus I_\pm(A)$

event set of A

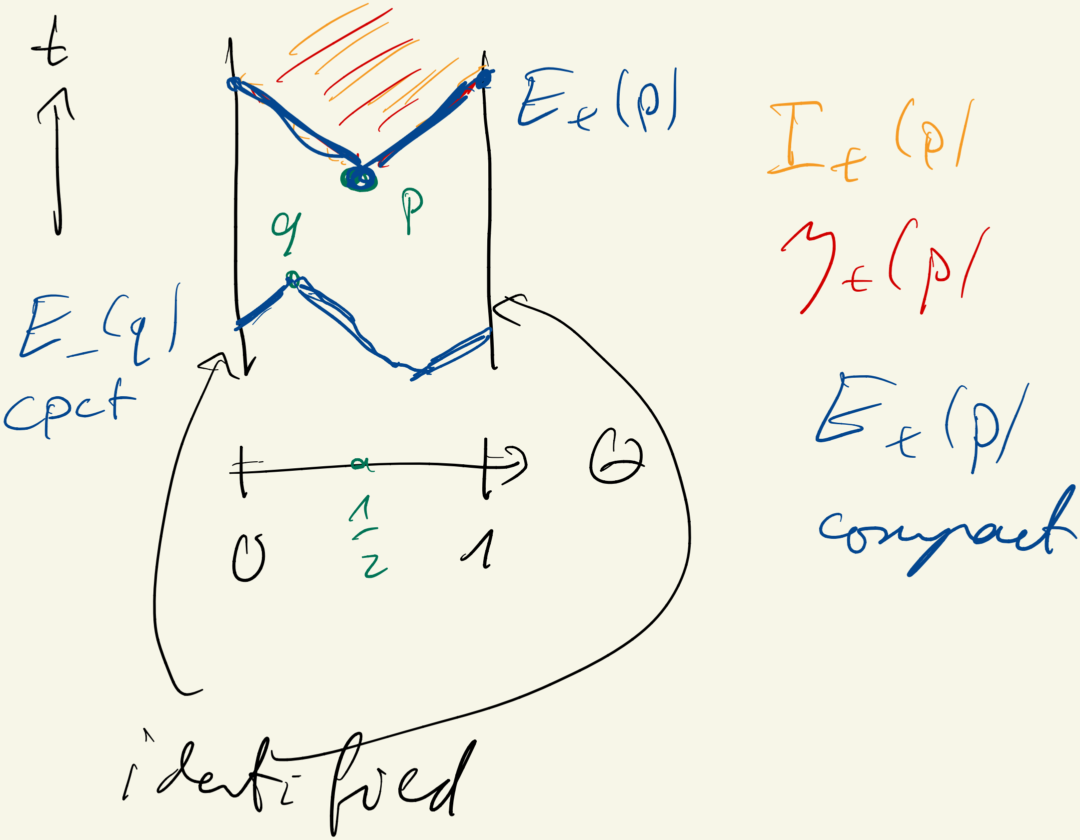
Def 2.118 A subset $A \subset M$ is called future-trapped if
past

- A is achronal
- A is closed
- if $\boxed{E_+(A) = J_+(A) \setminus I_+(A)}$

$E_-(A)$ is compact.

Example $V(g) = (\mathbb{R} \times S^1 \,|\, -dt^2 + d\theta^2)$

$$= \mathbb{R}^{1,1} / \mathbb{Z} e_1$$



$A = \{p\}$ is future trapped and past-trapped

Remarks

1.) A achronal $\Leftrightarrow A \cap I_t(A) = \emptyset$

$$\Rightarrow \underset{\text{closed}}{A} \subset \underbrace{J_t(A) \setminus I_t(A)}_{\text{cpct}} = E_t(A)$$

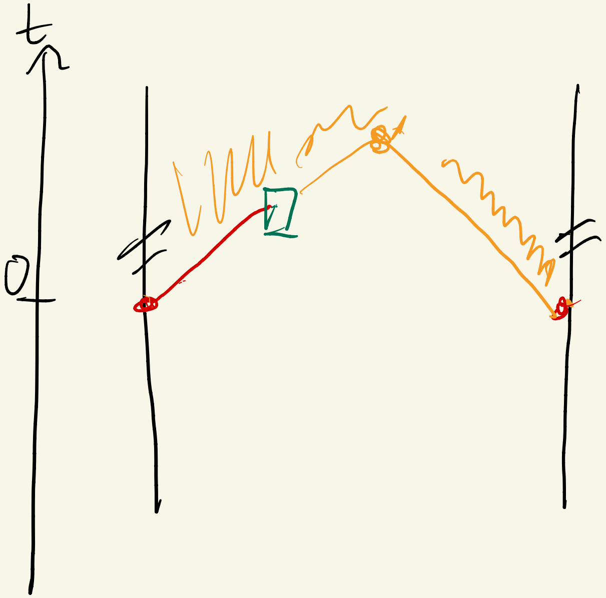
$$\Rightarrow A \text{ cpct}$$

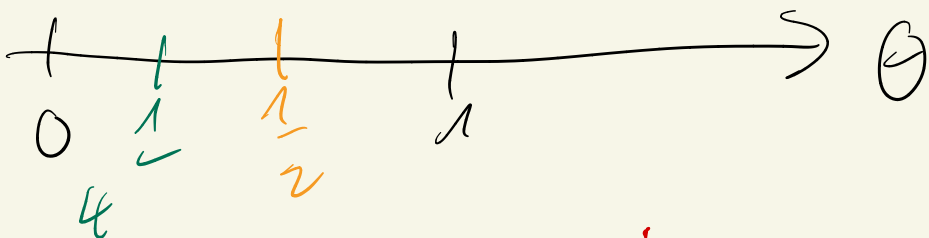
Thus A trapped $\Rightarrow A$ compact.

1a.) We cannot remove "closedness" in Def 2.118 without changing the defn.

$$M = \mathbb{R}^{1,1} / \mathbb{Z}e_1$$

$$A = \left\{ \begin{pmatrix} t \\ t \bmod \mathbb{Z} \end{pmatrix} \middle| \; 0 \le t < \frac{1}{4} \right\} \subset M$$





$$\mathcal{J}_t(A) = \mathcal{J}_t(\bar{A}) = \left\{ \begin{pmatrix} t \\ x \bmod \mathbb{Z} \end{pmatrix} \,\middle|\, t \geq |x|,\ -\frac{1}{2} \leq x \leq \frac{1}{2} \right\}$$

$$I_t(A) = I_t(\bar{A}) = \left\{ \begin{pmatrix} t \\ x \bmod \mathbb{Z} \end{pmatrix} \middle| \, t > |x|, \; -\frac{1}{2} \leq x \leq \frac{1}{2} \right\}$$

$$E_t(A) = E_t(\bar{A}) = \left\{ \begin{pmatrix} t \\ x \bmod \mathbb{Z} \end{pmatrix} \middle| \, t = |x|, \; -\frac{1}{2} \leq x \leq \frac{1}{2} \right\}$$

cpct

2.) For arbitrary $A \subset M$, the subset $E_t(A)$ is achronal:

$$p \in I_t(\underbrace{E_t(A)}_{\mathcal{J}_t(A)}) \subset I_t(\mathcal{J}_t(A))$$

$$= I_t(A) \Rightarrow p \notin E_t(A) = \mathcal{J}_t(A) \setminus I_t(A)$$

$$I_t(E_t(A)) \cap E_t(A) = \emptyset$$

$$\Rightarrow E_t(A) \text{ achronal.}$$

Lemma 2.121 $\quad m = n+1$

Let M be an m-dim Lorentzian mfd, $p \in M$, $\ell \in T_pM$ lightlike, and $e_1, \dots, e_{m-2}$ spacelike, orthonormal, $e_j \perp \ell$

Then $$\mathrm{ric}(\ell, \ell) = \sum_{j=1}^{m-2} \langle R(\ell, e_j)e_j, \ell \rangle$$

<u>Proof</u>: Assume $\ell = \lambda(e_0 + e_{m-1})$

$\lambda \in \mathbb{R}_{>0}$, $(e_0, e_1, \dots, e_{m-1})$ gen. orth. basis for T_pM

$\langle e_0, e_0 \rangle = -1$

$\ell, e_1, \dots, e_{m-2}$ basis for $\ell^\perp \subset T_pM$

$$0 = \langle R(\ell, \ell) e_0, \ell \rangle \qquad (1)$$

$$= \lambda (\langle R(\ell, e_0 + e_{m-1}) e_0, \ell \rangle)$$

$$0 = \lambda (\langle R(\ell, e_0 + e_{m-1}) e_{m-1}, \ell \rangle) \qquad (2)$$

$$0 = \frac{(1) - (2)}{\lambda} = \langle R(\ell, e_0) e_0, \ell \rangle$$

$$- \langle R(\ell, e_{m-1}) e_{m-1}, \ell \rangle$$

$\Longrightarrow$ Lemma

$$\langle R(\ell, X) Y, \ell \rangle = \langle R(\ell, Y) X, \ell \rangle$$

Prop 2.122 Let M be a Lorentzian mfd, $P \subset M$ a spacelike submfd of codim 2 and mean curvature vector field $\vec{H}$.

Let $c : [0, b] \to M$ be lightlike geodesic, $c(0) \in P$, $\dot{c}(0) \perp T_{c(0)} P$
$\underbrace{c(0)}_{p}$

Assume (i) $\operatorname{ric}(\dot{c}(t), \dot{c}(t)) \geq 0 \;\forall t \in [0, b]$

(ii) $\langle \vec{H}(p), \dot{c}(0) \rangle \geq \frac{1}{b}$

Then c has a focal point in $(0, b]$.

$p \in P$, $(T_p P)^{\perp} = N_p P \cong \mathbb{R}^{1,1}$

Proof: Assume that c has no focal point in $(0,b]$.

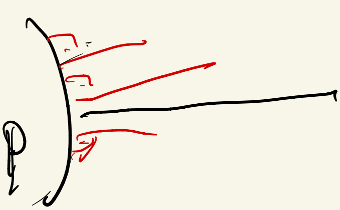
a) Let $e_1, \dots, e_{m-2}$ be an onb of T_pP. $J_i \in \Gamma(c^*TM)$ Jacobi field with

$i \in \{1, \dots, m-2\}$ $J_i(0) = e_i$,

$$\frac{\nabla J_i}{dt}(0) = - S_{\dot c(0)}(e_i)$$

$i = 0$ $\quad J_0(0) = 0, \quad \frac{\nabla J_0}{dt}(0) = \dot c(t)$

$$J_0(t) = t \cdot \dot c(t)$$



b) <u>Claim</u> $J_0(t), \ldots, J_{m-1}(t)$

is a basis of $(\dot{c}(t))^{\perp}$

for all $t \in (0, b]$.

$$\langle J_0(t), \dot{c}(t)\rangle = t \underbrace{\langle \dot{c}(t), \dot{c}(t)\rangle}_{=0} = 0$$

<u>$i > 0$</u>: $\frac{d^2}{dt^2}\langle J_i, \dot{c}\rangle = \langle \frac{D^2}{dt^2} J_i, \dot{c}\rangle$

$$= \langle R(\dot{c}, J_i)\dot{c}, \dot{c}\rangle = 0$$

$$\frac{d}{dt}\langle J_i(t), \dot{c}(t)\rangle\Big|_{t=0} = \langle \frac{D}{dt} J_i, \dot{c}\rangle\Big|_{t=0}$$

$$= -\langle \underbrace{S_{\dot{c}(0)}(e_i)}_{\in T_pP}, \underbrace{\dot{c}(0)}_{\perp T_pP}\rangle = 0$$

$$\langle y_i(0), \dot{c}(0) \rangle = \langle e_i, \dot{c}(0) \rangle = 0$$

$$\Rightarrow \quad y_i(t) \perp \dot{c}(t) \quad \forall t.$$

<u>Lin. independence</u> of $y_0(t), \dots, y_{m-2}(t)$

Assume $t_0 \in (0, \delta]$

$$\sum_{i=0}^{m-2} \alpha_i \, y_i(t_0) = 0, \quad \alpha_i \in \mathbb{R}$$

not all are 0.

$$y := \sum_{i=0}^{m-2} \alpha_i \, y_i, \quad y(t_0) = 0$$

$$y(0) = \sum_{i=1}^{m-2} \alpha_i \, e_i \in T_p P$$

$$\pi^{\tan}\left(\frac{Dy}{dt}(0)\right)$$

$$= \pi^{\tan}\left(\sum_{i=0}^{m-2} \alpha_i \frac{Dy_i}{dt}(0)\right)$$

$$= \pi^{\tan}\left(\sum_{i=1}^{m-2} \alpha_i (-1) S_{\dot{c}(0)}(e_i)\right)$$

$$= -\pi^{\tan}\left(S_{\dot{c}(0)}\left(\underbrace{\sum_{i=1}^{m-2} \alpha_i e_i}_{= y(0)}\right)\right)$$

$$= -\pi^{\tan}\left(S_{\dot{c}(0)}(y(0))\right)$$

$y \not\equiv 0$ t_0 is a focal point $\cancel{\&}$

There are $y_0(t), \ldots, y_{m-2}(t)$ lin. ind.