Theorem 2.112 (Hawking's singularity theorem)

Assume (M, g) is a time-oriented Lorentzian mfd with SEC.

Suppose S is a spacelike hypersurface and a Cauchy hypersurface with futr dir unit normal ν with $\langle \vec{H}, \nu \rangle \geq \beta$ ($\langle \vec{H}, \nu \rangle \leq -\beta$), $\beta > 0$.

Then every fut. dir. (past dir.) timelike curve starting in S has proper time bounded by $\frac{1}{\beta}$.

Interpretation

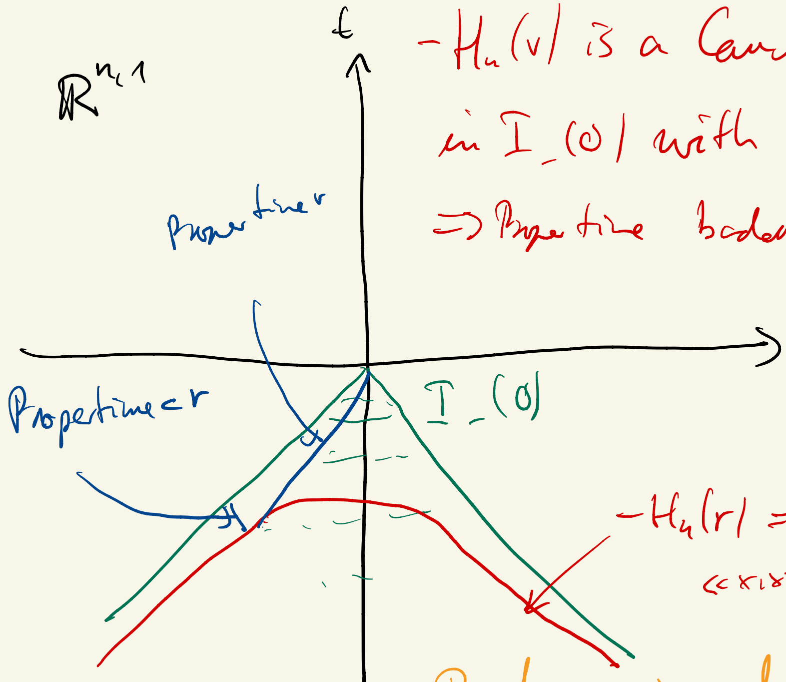
Does this predict a big crunch / a big bang?

1) **Pro** In a Robertson-Walker spacetime

$(0,b) \times_w N^n$, $-dt^2 + w(t)^2 g^N$ with $\mathrm{Ric}(\gamma',\gamma') \geq 0$ $\forall \gamma$ timelike

$w'(t_0) \leq -\beta$ we have seen that f.d. timelike curves starting in $\{t_0\} \times N$ have proper time $\leq \frac{1}{\beta}$. The space will then complete contract. $(w \searrow 0)$.

2) **Contra** $\mathbb{R}^{n,1}$



$-H_n(r)$ is a Cauchy hypersurface in $I_-(0)$ with $\langle H, v\rangle \geq \frac{1}{r}$

$\Rightarrow$ Proper time bounded by r

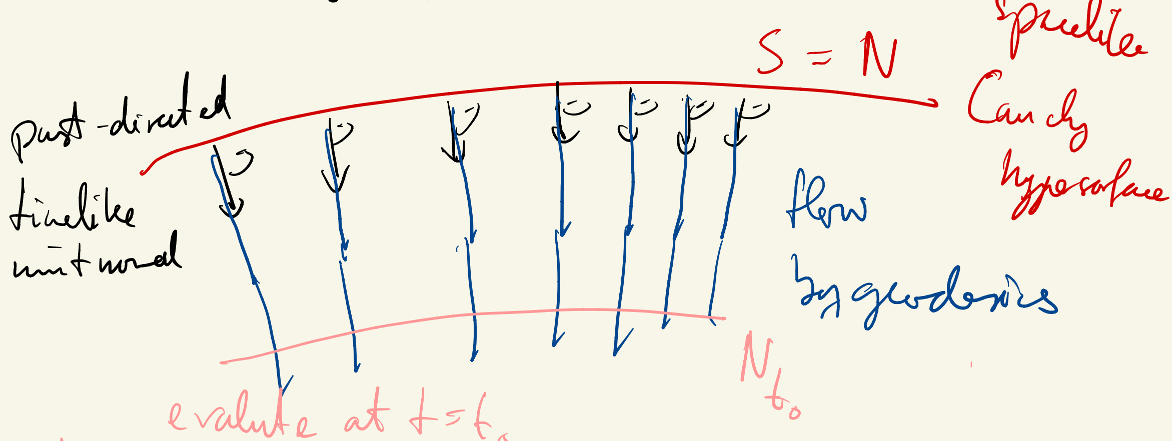
$I_-(0) \cong (0,\infty) \times H^n(r)$
$-dr^2 + r^2 g_{hyp}$

$-H_n(r) = \{x \in \mathbb{R}^{n,1} \mid \langle\langle x,x\rangle\rangle = -r^2, x^0 < 0\}$

But no singularity at 0!

The situation 2.) is unphysical

- Observation shows that our universe is close to an expanding Robertson-Walker spacetime.
- Refining the Hawking singularity theorem, one can show:



$\varphi_{t_0} : N \to N_{t_0}$ is a volume contracting map

The spacetime will be forced to shrink to a point.

- When the spacelike hypersurface shrinks, then matter accumulates strongly

Einstein equations $\Rightarrow$ curvature blow up not later than β

Space cannot be extended.

$$\underbrace{\operatorname{Ric} g - \frac{1}{2} \operatorname{Scal} \cdot g}_{\Rightarrow \text{ unbounded}} = c \underbrace{T(\cdot, \cdot)}_{\text{energy momentum tensor}}$$

$\Rightarrow$ see unbounded when we approach the „big bang".

diverges for matter concentration