Lecture no. 24

(3.)2.8 Hawking singularity theorem

<u>Def.</u>: A Lorentz manifold (M, g) satisfies the <u>strong energy condition</u> (SEC) if $\mathrm{Ric}(X,X) \geq 0$ for all timelike (causal) vectors X. (Assuming the Einstein equations hold and assuming $\underbrace{\Lambda = 0}$, this is equivalent to the defn in 3.1)

cosmological constant
= dark energy
appears in Einstein's equation

<u>Physics</u>: Our universe is supposed to satisfy the SEC.

<u>Example 3.2.114</u> Robertson-Walker spacetime

$$M = \underset{t}{(a,b)} \times N \qquad w \in C^\infty((a,b); \mathbb{R})$$

with $g = -dt^2 + (w(t))^2 g^N$ $\qquad n = \dim N = m-1$, $m = \dim M$

<u>Theorem 2.5.14 / Ex 2.5.9 cont'd</u>

$$\mathrm{ric}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -n \frac{w''(t)}{w(t)}$$

Strong energy cond. $\Rightarrow$ $w''(t) \leq 0$

$$\Rightarrow \forall t \geq t_0 \quad w'(t) \leq w'(t_0)$$

After Def 2.5.7

$$\underbrace{\langle \vec{\mathrm{II}}(u,v), \frac{\partial}{\partial t} \rangle}_{=: \mathrm{II}(u,v)} = -\frac{w'(t)}{w(t)}\, g(u,v)$$

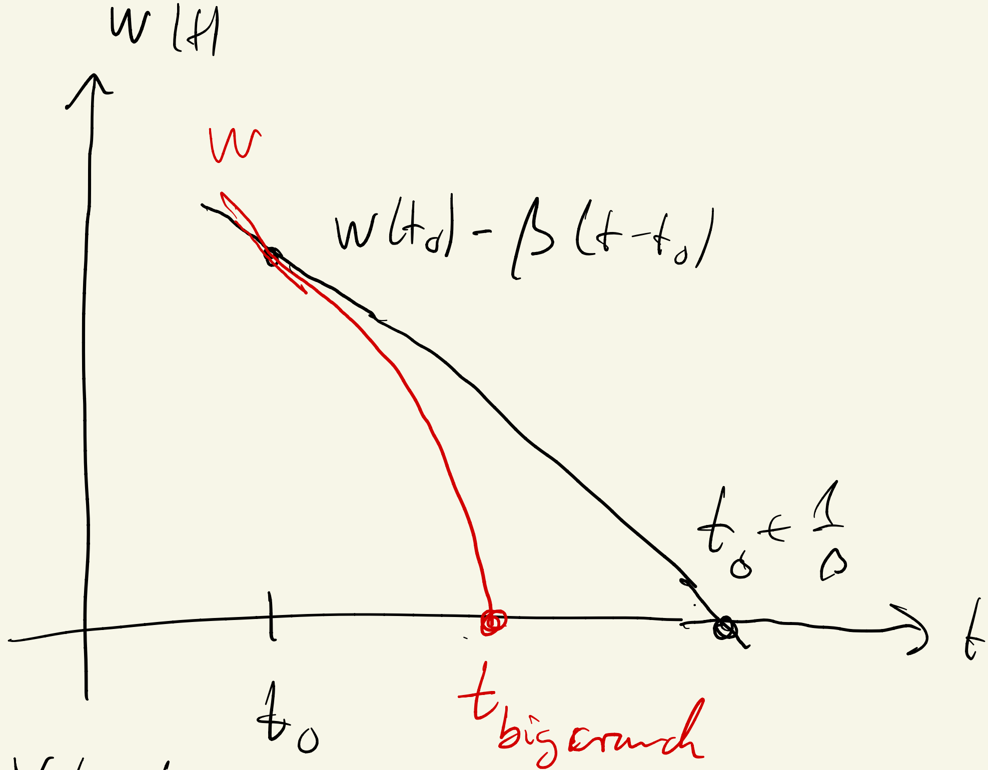
where $u,v \in T_p N$

$$H = \frac{1}{n} \mathrm{tr}_g \mathrm{II} = -\frac{w'(t)}{w(t)} \frac{1}{n} \underbrace{\mathrm{tr}_g g}_{n}$$

$$= -\frac{w'(t)}{w(t)}$$

Thus if $H|_{t_0} = \langle \vec{H}, \frac{\partial}{\partial t} \rangle|_{t=t_0} \geq \beta > 0$

then $w'(t_0) \leq -\beta\, w(t_0)$



$\forall t \geq t_0$

$$w'(t) \leq -\beta w(t_0) \qquad w : (a, b) \to \mathbb{R}$$

If b is sufficiently large then

$\lim w(t) = 0$ for some

$t \nearrow t_{bigcrunch}$

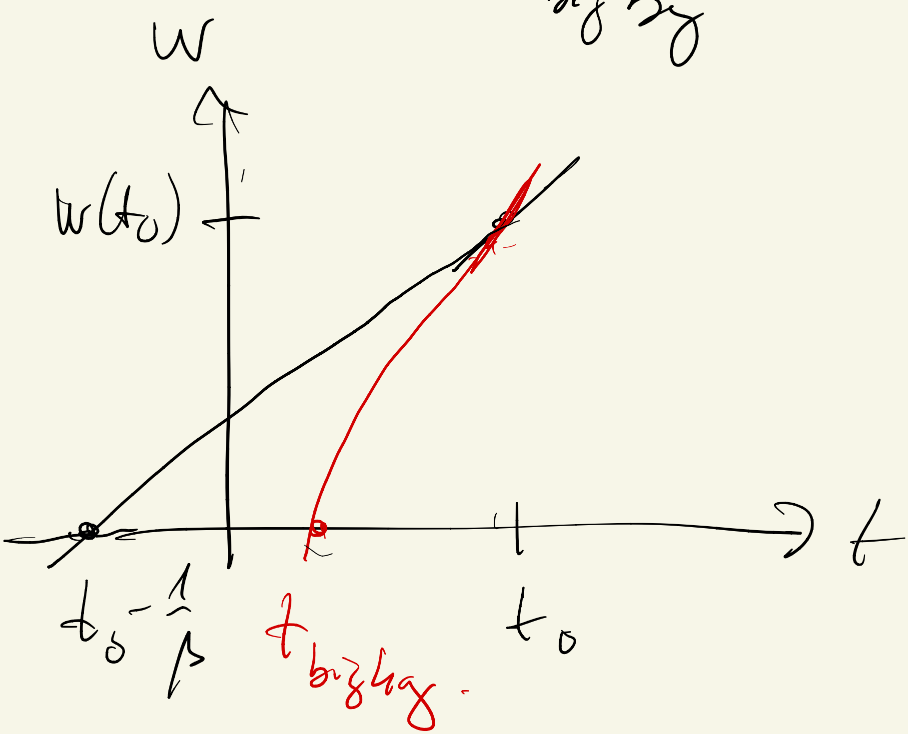
$$t_{bigcrunch} \in (t_0, t_0 + \frac{1}{\beta}].$$

Similarly If SEC holds and

$H \leq -\beta < 0$, then for

some $t_{\text{big bag}} \in [t_0 - \frac{1}{\beta}, t_0)$

we have $\lim\limits_{t \to t_{\text{big bag}}} w(t) = 0$



Physical observation: Redshift

$\rightsquigarrow$ Hubble's law

A galaxy / star with distance x to us increases its distance to us by βx where

$$\beta_{phy} = 70 \frac{km}{s} \frac{1}{Mpc} \qquad 1 MPc \approx 3 \cdot 10^{19} km$$

$$= \underset{\underset{\mathbb{R}}{\cap}}{(-\beta)} \frac{1}{s}$$

$\to \beta$ Hubble constant

$$\beta < 0$$

<u>Theorem 2.112</u> (Hawking's singularity theorem)

Assume (M, g) is a time-oriented Lorentzian mfld with SEC.

Suppose S is a spacelike hypersurface and a Cauchy hypersurface with futr. dir unit normal ν with $\langle \vec{H}, \nu \rangle \geq \beta$ $(\langle \vec{H}, \nu \rangle \leq -\beta)$, $\beta > 0$.

Then every fut. dir. (past dir.) timelike curve starting in S has proper time bounded by $\frac{1}{\beta}$.

Prop (Lor):

Let (M,g) be a time-oriented Lorentzian mfd with SEC and let S be a Cauchy hypersurface (in part. $M = I_+(S) \mathbin{\dot\cup} S \mathbin{\dot\cup} I_-(S)$).

Then for any $p \in I_+(S)$, there is a future-directed time-like curve c (of maximal proper time) from S to p, i.e. $c : [a,b] \to M$ f.d. time like, $q = c(a) \in S$, $c(b) = p$

$$L(c) = \sup_{q \in S} \underbrace{\tau(q,p)}_{\text{time separation}}$$

$$\tau(q,p) = \sup \{ L(\tilde{c}) \mid \tilde{c} \text{ f.d.}$$

timelike piecewise C^1-curves

from q to p $\}$

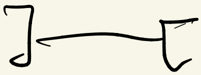
time-separation

<u>Prf of Prop</u> E.g. in Bär Lorentzian geometry 2.6 & 2.7

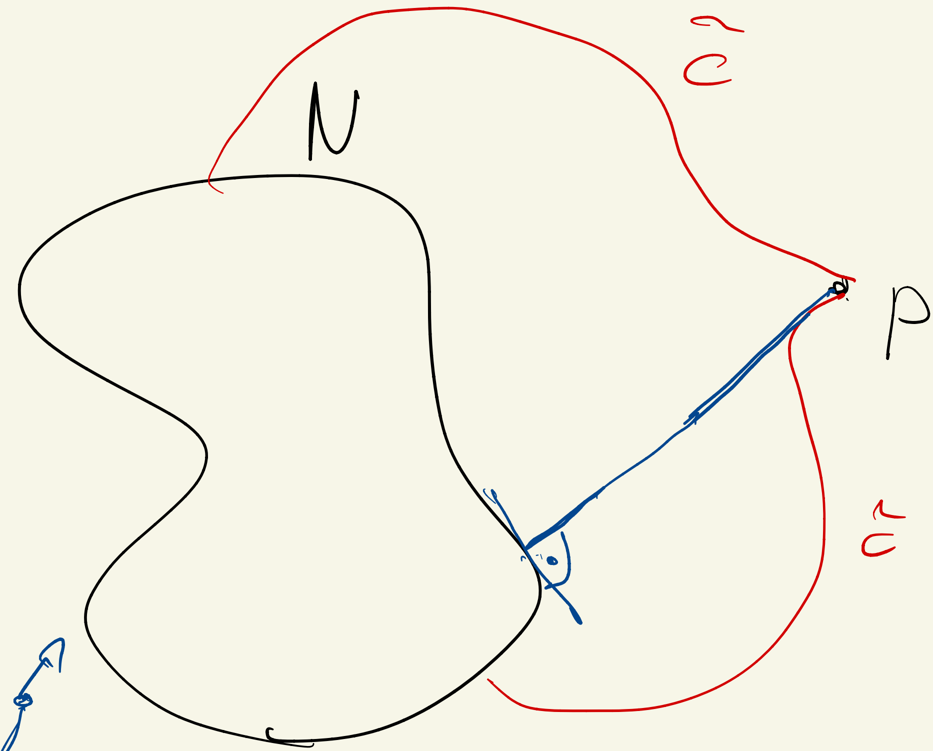
Prop (Riem.)

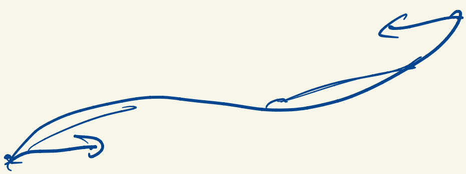
Let N be a closed (as a top. subspace) submfd of a complete connected Riem. mfd M. Then for any $p \in M$ there is a piecewise C^1-curve $c : [a,b] \to M$, $c(a) \in N$, $c(b) = p$ such that $L(c) = d(p, N) :\overset{\text{def}}{=} \inf \{ L(\hat{c}) \mid \hat{c}$ curve from some $q \in N$ to $p \}$

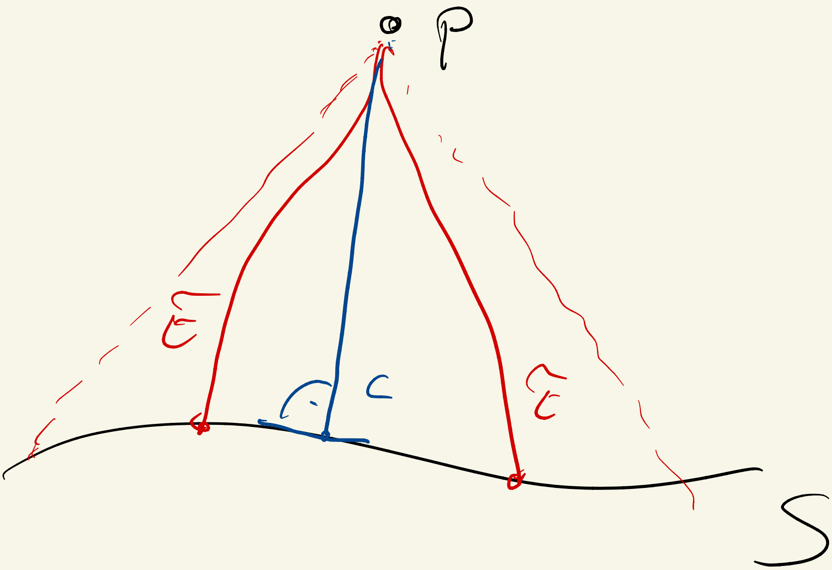
$(a,b] \times \{0\} \subset \mathbb{R}^2$



not closed







P
ε
c
ε
S

Proof of Thm 2.112

Let $\tilde{c}$ be f.d. timelike curve from S to some $p \in M$

$$\Rightarrow \sup_{q \in S} \tau(q,p) \geq L(\tilde{c})$$

Let c be a curve of max proper time from S to p, given by Prop (Lor.), c parametrized by proper time, $c : [0,b] \to M$.

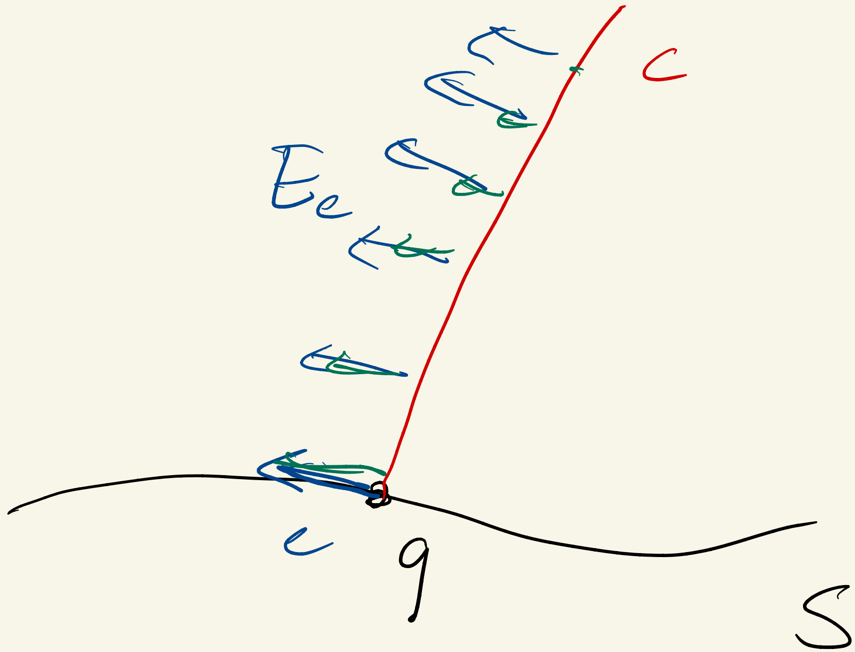
$q := c(0) \in S$.

Then c is a timelike geodesic (see recap of lect. 19) and $\dot{c}(0) \perp T_q S$.

<u>To show</u> $b = \mathcal{L}[c] \leq \frac{1}{\beta}$

$$\left(\Rightarrow \mathcal{L}[\hat{c}] \leq \mathcal{L}[\mathring{c}] \leq \frac{1}{\beta} \right)$$

For $e \in T_q S$, $\|e\| = 1$, define $E_e \in \Gamma(c^* TM)$ as the parallel vector field with $E_e(0) = e$. $V_e(t) := \left(1 - \frac{t}{b}\right) E_e(t)$



c
E_e
e
q
S

$\rightsquigarrow V_e(t) \perp \dot{c}(t)$

Let $c_s : [0,b] \times (-\varepsilon,\varepsilon) \to M$ be a variation of c with variation v.f. V_e:

$$\frac{\nabla}{dt} V_e = -\frac{1}{b} E_e \perp \dot{c}$$

$$\underbrace{\frac{\partial^2}{\partial s^2}\Big|_{s=0} L[c_s]}_{0 \,\leq} = \langle A(0), \dot{c}(0)\rangle - 0$$

$$- \int_0^b \langle R(E_e, \dot{c}) E_e, \dot{c}\rangle \left(1-\frac{t}{b}\right)^2 dt - \int_0^b \left\langle -\frac{1}{b} E_e, -\frac{1}{b} E_e \right\rangle dt$$

$$= \Big\langle \underbrace{\frac{\nabla}{ds}\Big|_{s=0} \frac{\partial c_s}{\partial s}}_{= \vec{\mathrm{II}}(V_e, V_e)}, \dot{c}(0) \Big\rangle$$

$$+ \int_0^b \left(1 - \frac{t}{b}\right)^2 \langle R(E_e, \dot{c})\dot{c}, E_e \rangle \, dt$$

$$-\frac{1}{b} = \langle \vec{\mathrm{II}}(V_e, V_e), \dot{c}(0) \rangle - \frac{1}{b} + \int_0^b \left(1 - \frac{t}{b}\right)^2 \dots$$

Now additionally assume

$$\frac{\nabla}{ds}\Big|_{s=0} \frac{\partial c_s}{\partial s} \perp T_{c(0)} S$$

(we choose c_s s.th. $s \mapsto c_s(b)$ is a geodesic in S)

Now choose a onb $e_1, \dots, e_n$ $(n = m-1 = \dim S)$ of $T_{c(0)} S$

Let e run through this basis

sum over all e_i.

$$0 \geq \sum_{j=1}^{n} \langle \vec{\Pi}(\underset{e_j}{V_{e_j}}, \underset{e_j}{V_{e_j}}), \overbrace{\dot{c}(0)}^{U \text{ normal vect.}} \rangle$$

$$\underset{a}{\mathrm{ric}}(\dot{c}, \dot{c}) \geq 0$$

$$+ \int_0^b \left(1 - \frac{t}{b}\right)^2 \left(\sum_{j=1}^{n} \langle R(E_{e_j}, \dot{c})\dot{c}, E_{e_j} \rangle + \langle R(\dot{c}, \dot{c})\dot{c}, \dot{c} \rangle \right) dt$$

$$-\frac{n}{b} \geq n \langle \vec{H}, U \rangle - \frac{n}{b}$$

$$\Rightarrow \langle \vec{H}, \nu \rangle \leq \frac{1}{b}$$

$$\leq \beta$$

$$\Rightarrow b \leq \frac{1}{\beta}$$

$\square$

<u>Attention</u> Example

$M = \mathbb{R}^{n,1} \quad n = m-1$

$S := -H^n(r) = \{ x \in \mathbb{R}^{n,1} \mid$

$\langle\langle x, x \rangle\rangle = -r^2, \; x^0 < 0 \}$

Then $\nu(p) = -\frac{p}{r}$ is f.d. unit normal v. f for S.

$$S_\nu(X) = -\nabla_X \nu = \frac{1}{r} \cdot X$$

$$\Rightarrow \vec{\mathrm{II}}(X,Y) = -\frac{1}{r} \langle\langle X,Y \rangle\rangle \cdot \nu$$

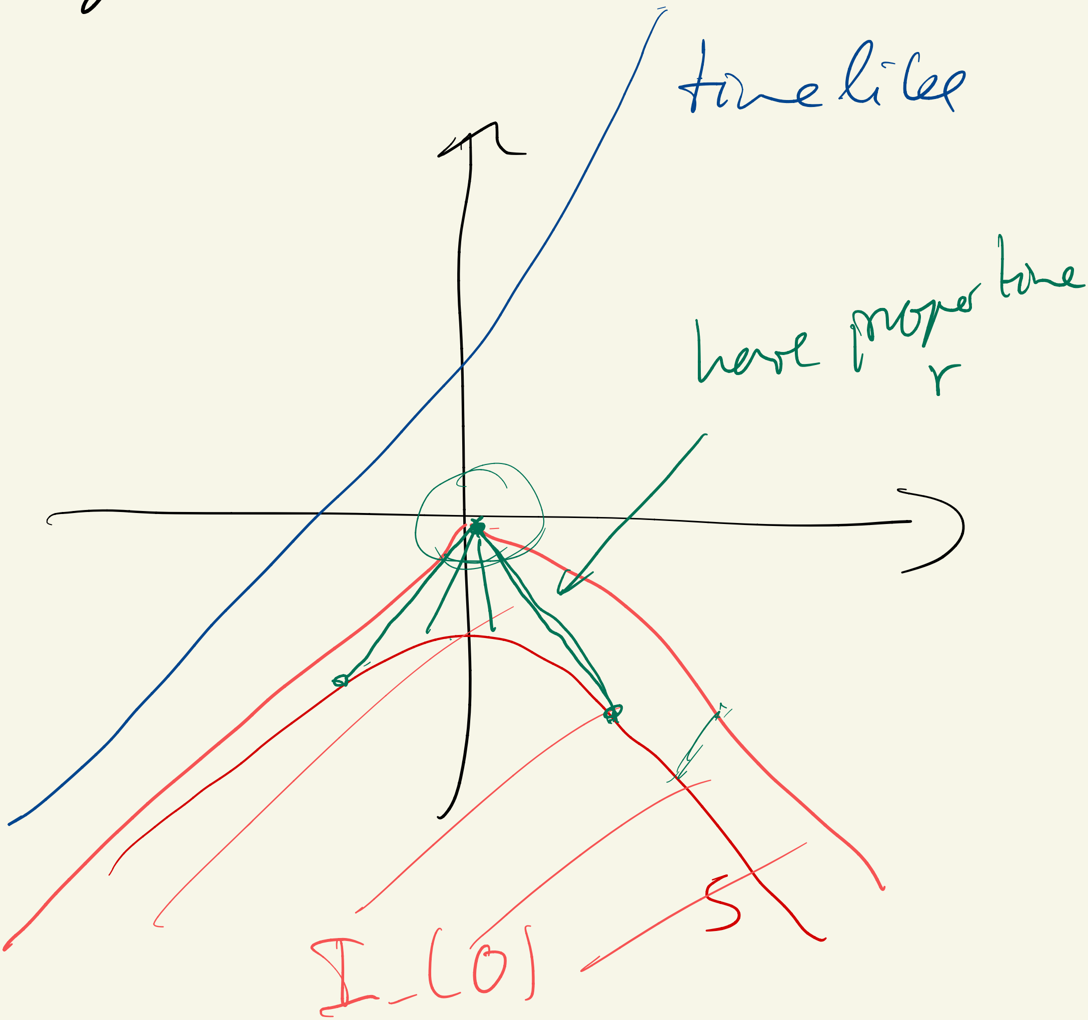
$$\Rightarrow \vec{H}(p) = -\frac{p}{r^2}$$

$$\langle\langle \vec{H}, \nu \rangle\rangle = \beta = \frac{1}{r}$$

But every fut. directed timelike geodesic starts in S has infinite proper time.

Is this a contradiction?

No: S is <u>not</u> a Cauchy hypersurface of $\mathbb{R}^{4,1}$!



But S is a Cauchy hyper-surface of $I_-(0) \subset \mathbb{R}^{4,1}$