Recap from lecture no. 19 M Lorentzian

Let $c_\bullet : [a,b] \times (-\varepsilon,\varepsilon) \to M$ be a variation of a timelike smooth curve $c : [a,b] \to M$, $c = c_0$, parametrized by proper time.

1st variation formula (Prop 2.44a)

$$\frac{\partial}{\partial s}\Big|_{s=0} \mathcal{L}[c_s] = \int_a^b \langle V, \frac{\nabla}{dt}\dot{c} \rangle \, dt$$

$$+ \langle V(a), \dot{c}(a) \rangle - \langle V(b), \dot{c}(b) \rangle$$

where $V(t) := \frac{\partial c_s}{\partial s}(t,0)$.

Thus if c is a curve of maximal proper time from S to p, then $\frac{\nabla}{dt}\dot{c} = 0$ ($\Leftrightarrow$ c is a timelike geodesic) and $\dot{c}(a) \perp T_{c(a)}S$.

2^{nd} variation formula

Additionally assume c is a timelike geodesic.

$$A(t) := \frac{D}{ds}\Big|_{s=0} \frac{\partial c_s}{\partial s} \quad \text{acceleration}$$

$$\frac{\partial^2}{\partial s^2}\Big|_{s=0} \mathcal{L}[c_s] = \langle A(a), \dot{c}(a) \rangle - \langle A(b), \dot{c}(b) \rangle$$

$$- \int_a^b \langle R(V, \dot{c}) V, \dot{c} \rangle \, dt$$

$$- \int_a^b \langle \pi^{nor}\left(\frac{DV}{dt}\right), \pi^{nor}\left(\frac{DV}{dt}\right) \rangle \, dt$$