Schwarzschild spacetime

Mittwoch, 30. Juni 2021 05:56

Schwarzschild Spacetime

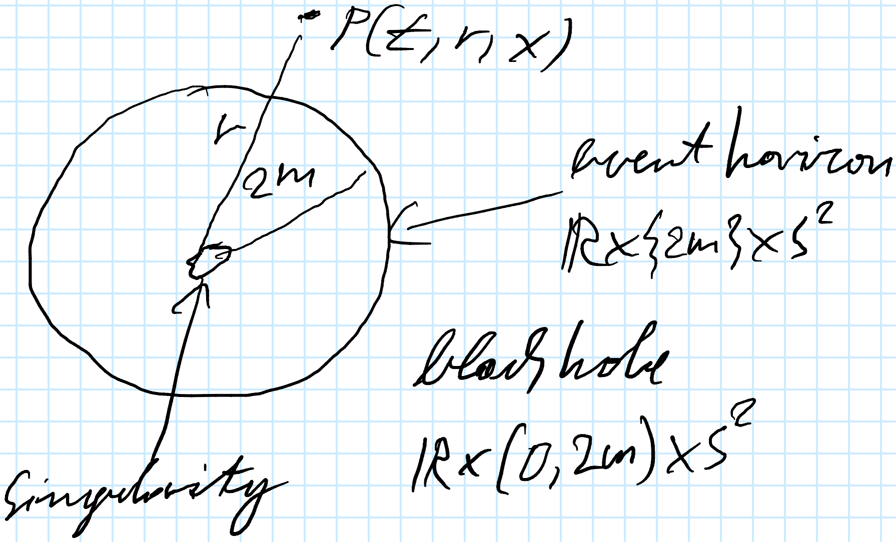
$$g = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2 g_{sph}$$

on $\mathbb{R} \times [(0,2m) \cup (2m,\infty)] \times S^2$

It holds (see ex. sheets):

- g is analytic
- Isotropy: $\forall A \in O(3)$:
  $(x^0, \vec{x}) \mapsto (x^0, A\vec{x})$ is an isometry
- g is static: $\forall t_0 \in \mathbb{R}$:
  $(x^0, \vec{x}) \mapsto (\pm x^0 + t_0, \vec{x})$ is an isometry
- asymptotically Minkowski, i.e.
  $g \to g_{mink}$ for $r \to \infty$
- ricci flat $\Rightarrow$ vacuum solution
- globally hyperbolic

"Solution of the most obvious/desirable choices" See Bär RT p66–70 for derivations.



$2m$ is called the Schwarzschild radius.

Uniqueness (Jebsen): Any spatially spherically sym. vacuum solution of the EFEq is isometric to an open subset of a Schwarzschild metric.

Spatially spherically sym.: $\Leftrightarrow$ globally hyperbolic and with a faithful action of $SO(3)$ with all orbits diffeo. to S^2 and grad of $(p \mapsto \mathrm{vol}(SO(3)\cdot p))$ is spacelike.

"$(p \mapsto \mathrm{vol}(SO(3)\cdot p))$" is spacelike.

Def.: A Schwarzschild observer is a curve $\gamma(s) = (t(s), r_0, x_0)$ parametrized by proper time.

$$-1 = g(\dot\gamma, \dot\gamma) = (t')^2 g(\partial_t, \partial_t)$$
$$= -(t')^2 \left(1 - \frac{2m}{r_0}\right)$$
$$\Rightarrow \underline{t' = \frac{1}{\sqrt{1-\frac{2m}{r_0}}}} \quad \text{if } \gamma \text{ is f.d.}$$

$$\Rightarrow \gamma(s) = \left(t_0 + \frac{s}{\sqrt{1-\frac{2m}{r_0}}}, r_0, x_0\right)$$

Acceleration of observer?

$$\frac{\nabla}{ds}\dot\gamma = \nabla_{t'\partial_t}(t'\partial_t)$$
$$= \frac{1}{1-\frac{2m}{r_0}} \frac{(r_0-2m)m}{r_0^2} \partial_r$$
$$= \frac{m}{r_0^2}\partial_r$$

$$\left\| \frac{m}{r_0^2} \partial_r \right\| = \sqrt{g(\cdot,\cdot)} = \frac{m}{r_0^2} \frac{1}{\sqrt{1-\frac{2m}{r_0}}} \to \frac{m}{r_0^2}$$

for $r_0 \to \infty$

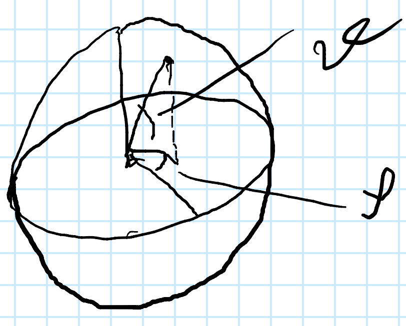
Comparison with Newtonian physics justifies to call m the mass of the object in the center.

$\mathcal{L}_X g = 0$ Killing vector field

Schwarzschild spacetime $=: M$

∂_t is Killing, because M is static

∂_φ is Killing, because of sperical sym.



Let $\gamma(s) = (t(s), r(s), \varphi(s), \vartheta)$

be a geodesic and $\vartheta \equiv \frac{\pi}{2}$

(possible because

(possible because
$(t, r, \vartheta, \varphi) \mapsto (-t, r, \pi - \vartheta, \varphi)$
is an isometry)

$\dot{\gamma} = t' \partial_t + r' \partial_r + \varphi' \partial_\varphi$

Sheet 5 ex 1 a)

$\left(s \mapsto \langle \dot{\gamma}(s), K_{\gamma(s)} \rangle\right) \equiv$ const.

$E := \langle \dot{\gamma}, \partial_t \rangle$ (energy)

$L := \langle \dot{\gamma}, \partial_\varphi \rangle$ (angular momentum)

These are constants.

$E = -t' h \quad , \quad L = \varphi' r^2 \quad ,$

$h = 1 - \frac{2m}{r}$

If γ is lightlike

$0 = g(\dot{\gamma}, \dot{\gamma}) = -(t')^2 h + \frac{(r')^2}{h} + r^2 (\varphi')^2$

$\quad = t' E + \frac{(r')^2}{h} + \varphi' L$

$= t L + \frac{\;}{h} + \varphi L$

$\Rightarrow$ Energy eq. for light particles:

$$E^2 = (r')^2 + \varphi' L h = (r')^2 + \frac{L^2}{r^2} h$$

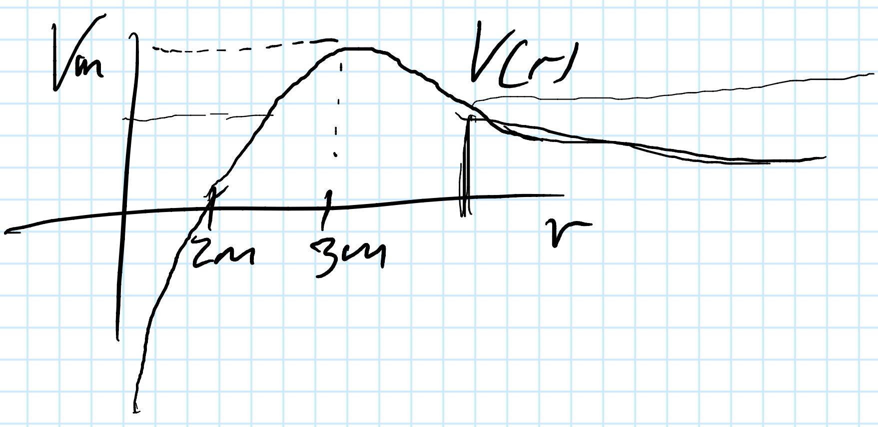
A similar energy eq. exists for massive particles (i.e. timelike).

Trajectories of light particles ($L \neq 0$):

$$V(r) := \frac{L^2}{r^2} h(r)$$

$$V(2m) = 0\,, \quad V(r) \to \begin{cases} 0 & \text{for } r \to \infty \\ -\infty & \text{for } r \to 0 \end{cases}$$

$$V'(r) = \dots = \frac{2L^2}{r^4}(-r + 3m)$$



$$E^2 = \underline{(r')^2} + V(r) \Rightarrow V(r) \leq E^2$$

≥ 0

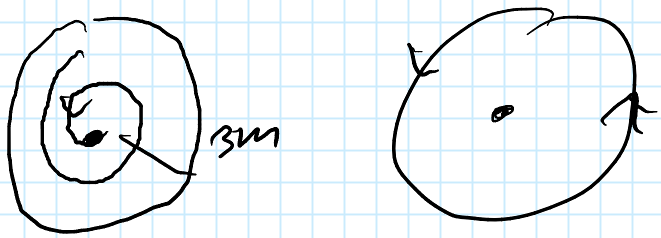
Case 1 ($E^2 < V_m$):

a)

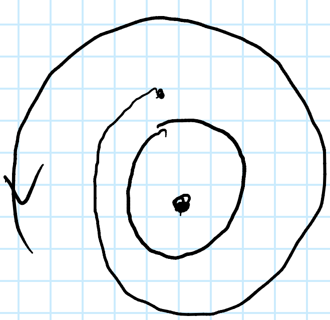
b)

Case 2 ($E^2 = V_m$)

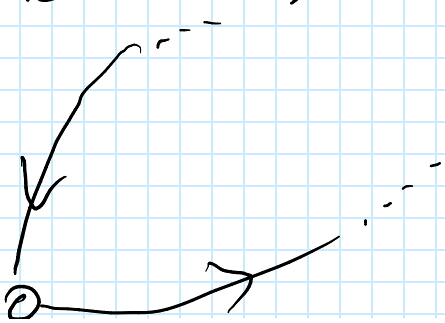
$r_0 < 3m$ $\qquad$ $r_0 = 3m$

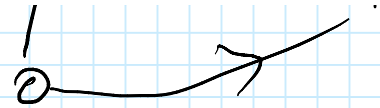


$r_0 > 3m$



Case 3 ($E^2 \geq V_m$):





Massive particles analogous (see O'Neil chap. 13 for an extensive treatment)

Finkelstein diagram

For an F with $F'(r) = \left(1-\frac{2m}{r}\right)^{-1}$ pullback g with

$$\phi : \mathbb{R}\times(2m,\infty)\times S^2 \circlearrowleft$$
$$(v,r,x) \mapsto (v-F(r), r, x)$$

and extend it analytically to $\mathbb{R}\times\mathbb{R}_{>0}\times S^2$ (sheet 7 ex. 3)

The Finkelstein diagram shows the new situation on $(\mathbb{R}\times\mathbb{R}_{>0}\times S^2, \phi^* g)$.

$\Rightarrow$ no real singularity at $r=2m$ but a bad chart

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$r = 0$ is a real singularity, because $R(\cdot,\cdot)\cdot$ diverges there.

Same is possible for

$$\tilde{\phi}(w, r, x) := (w + F(r), r, x)$$

Call these new mfds M' and M'' with Lor. metric the extensions g' and g'' of $\phi^* g$ and $\tilde{\phi}^* g$.

$(M' = M'' = \mathbb{R} \times \mathbb{R}_{>0} \times S^2)$

<u>Kruskal diagram</u>

There exist isometries that embed (M', g') and (M'', g'') into $\mathbb{R}^2 \times S^2$ and restrict on

$M \; (\cong M'_{r>2m} \subset M'$ resp. $\cong M''_{r>2m} \subset M'')$

to:

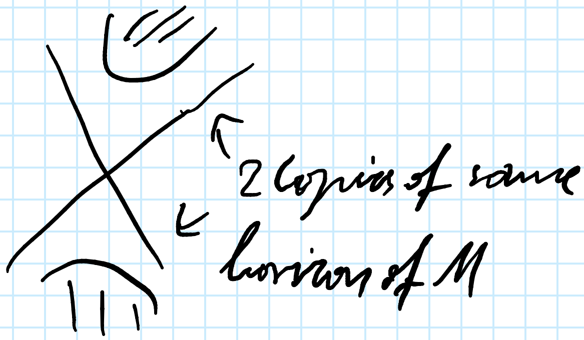
$x' = \frac{1}{2}(v' - w')$, $t' = \frac{1}{2}(v' + w')$, where

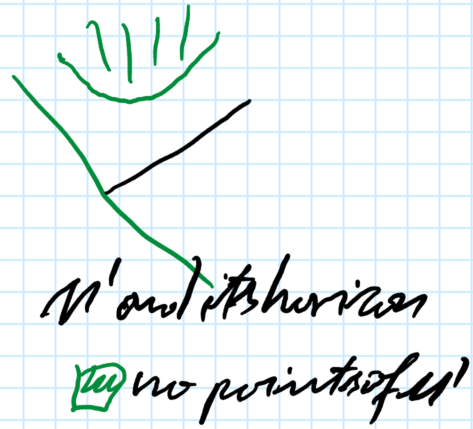
$v' = \exp(v/4m)$, $w' = -\exp(-w/4m)$.

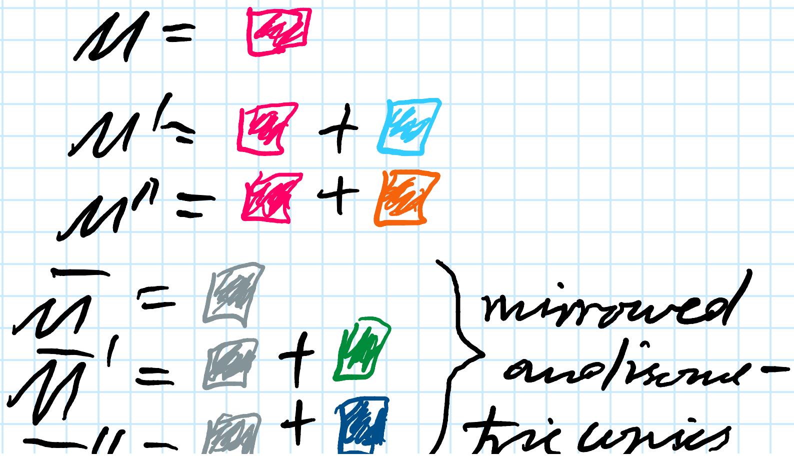
A glance at the Finkelstein diagram shows: For (v_0, r) with $r \to 2m$ one has $w \to \infty \Rightarrow v' = v_0' \wedge w' \to 0$.

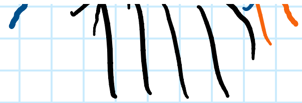
And the same holds for (w_0, r):
$w' = w_0' \wedge v' \to 0$.

So in the Kruskal diagram one has two copies of the same event horizon of M; that is $0 < x^1 = t^1$ (the one of M') and $0 < x^1 = -t^1$ (that of M'').









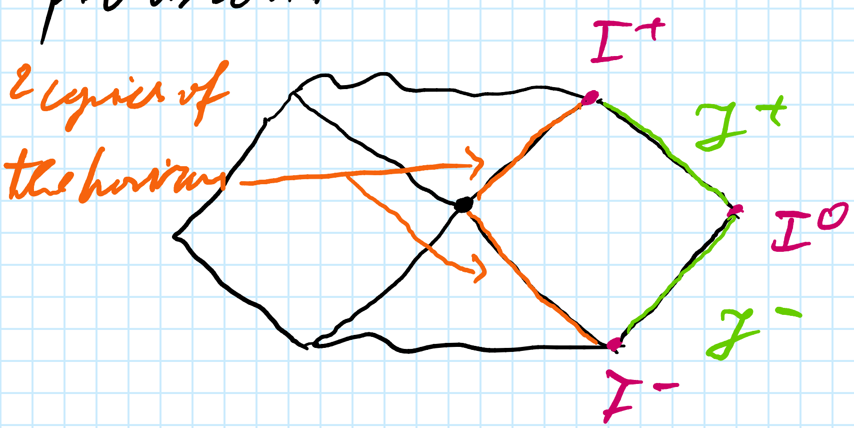
$M' = \square + \square$ } and isometric copies
$M'' = \square + \square$

The Penrose diagram can be obtained by computing x' and t' from

$$v'' = \arctan\left(\frac{v'}{\sqrt{2m}}\right) \text{ and}$$
$$w'' = \arctan\left(\frac{w'}{\sqrt{2m}}\right)$$

instead of v' and w' and then compactifying (taking the closure in the x'-t'-plane).

By asymptotic flatness $g \to g_{mink}$ and the same calculations as in ex. 3e) sheet 8 one gets the same results for what concerns the shape of the diagram and the "endpoints at infinity" of geodesics within this compactification:



$I^\pm$ from future- resp. past directed timelike geodesics timelike geodesics

I^0 from spacelike geodesics

$\mathcal{I}^{\pm}$ from future- resp. past directed lightlike geodesics.