# Robertson-Walker spacetime

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## Robertson-Walker Spacetimes

Historically and scientifically important solution of

$G + \Lambda g = \kappa T$ (Einstein field eq.)

$G = \mathrm{ric} - \frac{1}{2}\,\mathrm{scal}\cdot g$

Choose the physical units s.t.

$c = 1 =$ gravitational constant

$\Rightarrow \kappa = 8\pi$.

R-W. spacetimes: simple model of spacetime. Spacial part $(S, g_S)$

Riem mfd, connected, complete, loc. isotropic

One has:

$\dim S \geq 3$ and Schur's lemma and loc. isotropy

$\Rightarrow \mathrm{sec} = \mathrm{sec}_p \equiv \varepsilon \in \mathbb{R}$ constant

Def.: An $(n+1)$-dim. mfd $(M, g)$ is called a R-W spacetime if

$M = I \times S$, $g = -dt^2 + f^2 g_S$

where $I \subset \mathbb{R}$ intervall, $(S, g_S)$ conn., where $I \subset \mathbb{R}$ intervall, $(S, g_S)$ conn., complete, Riem mfd with sec $\equiv \varepsilon$ and $f$ smooth and positive.

Example:

- $(S, g_S) = (\mathbb{R}^n, g_{eucl}) \Rightarrow (M, g) = (\mathbb{R}^{n+1}, g_{mink})$

- sheet 12: $I_+(0) \subset \mathbb{R}^{n+1}$

Geodesics: let $x_0 \in S$, then

$\gamma(s) = (s, x_0)$ this is a geodesic.

Proof: let $\sigma: (x_0 \in) U (\subset S) \to U$ geodesic reflection about $x_0$

$\Rightarrow \phi: I \times U \circlearrowleft$, $(t, x_0) \mapsto (t, \sigma(x_0))$

is an isometry with $Fix(\phi) = I \times \{x_0\}$, which is parametrized by $\gamma$ by proper time $\square$

- let $\gamma(s) = (\gamma^0(s), \tilde{\gamma}(s))$ be a lightlike geodesic ($\tilde{\gamma}$ is a curve in $S$)

$0 = g(\dot{\gamma}, \dot{\gamma}) = g((\gamma^0)'(s) \frac{\partial}{\partial t} + \tilde{\gamma}, (\gamma^0)'(s) \frac{\partial}{\partial t} +$

$+ \tilde{\gamma})$

$$0 = g(\dot\gamma,\dot\gamma) = g((\gamma^0)'(s)\,\partial t + f\,\tilde{\gamma} \ldots$$

$$+ \tilde{\dot\gamma}]$$

$$= -(\dot\gamma^0)^2 + f^2(\gamma^0)\, g_S(\dot{\tilde\gamma}, \dot{\tilde\gamma}) \text{ and}$$

$$\partial_s\left(f(\gamma^0(s)) \cdot (\gamma^0)'(s)\right) =$$

$$= f'(\gamma^0(s))(\gamma^0)'^2(s) + f(\gamma^0)(\gamma^0)''(s)$$

$$= (f' \cdot f^2)(\gamma^0(s))\, g_S(\dot{\tilde\gamma}, \dot{\tilde\gamma}) + f(\gamma^0(s))(\gamma^0)''(s)$$

$= 0$ (follows from the geodesic eq in coordinates)

$$\Rightarrow f(\gamma^0(s))\,(\gamma^0)'(s) = \text{const.}$$

In physics: 4-momentum ($m_0 \neq 0$)

$$(p^\alpha) = \left(\frac{E}{c}, \vec{p}\right),\ c = 1,$$

$$p^\alpha = m_0 \frac{dx^\alpha}{d\tau} = \text{const.} \cdot \frac{dx^\alpha}{ds},\ \frac{d\tau}{ds} = \sqrt{g(\dot\gamma,\dot\gamma)}$$

If $\gamma(s)$ describes a worldline of a photon:

$$\frac{E(s_1)}{E(s_2)} = \frac{f(\gamma^0(s_2))}{f(\gamma^0(s_1))}$$

$$E = h \cdot \lambda^{-1} \Rightarrow z := \frac{\lambda(s_2) - \lambda(s_1)}{\lambda(s_1)} = \frac{f(\gamma^0(s_1))}{f(\gamma^0(s_2))} - 1$$

redshift of the photon / geodesic.

redshift of the proton/geodesic.

Set $H(t) := \frac{f'(t)}{f(t)}$ the Hubble constant

By Taylor expansion of $f$:

$$z = \frac{1}{f(t_2)}\left(f(t_2)+f'(t_2)(t_1-t_2)+\theta(|t_1-t_2|^2)\right)-1$$

$$\approx H(t_2)(t_2-t_1)$$

From Ex 2.5.15 (page 65) of the partial lecture notes:

$$G|_{TS\otimes TI \oplus TI\otimes TS} \equiv 0$$

From the physical meaning of $T$ define

$p := G|_{TS^{\otimes 2}}$ (isotropic) pressure

$\rho := G|_{TI^{\otimes 2}}$ mass density

From now on: $\dim S = 3$, $\varepsilon \in \{\pm 1; 0\}$

$(g \rightsquigarrow c^2 g \Rightarrow \sec \rightsquigarrow \frac{1}{c^2}\sec)$

Now one gets from Ex 2.5.15

$$\frac{8\pi}{3}\rho = \left(\frac{f'}{f}\right)^2 + \frac{\varepsilon}{f^2} \quad (1)$$

$$-8\pi p = \left(\frac{f'}{f}\right)^2 + \frac{\varepsilon}{f^2} + 2\frac{f''}{f} \quad (2)$$

## Singularities

Let the domain $(t_*, t^*)$ of $f$ be maximal (i.e. $f$ cannot be extended beyond as a pos. smooth fct.)

$$-\infty \leq t_* < t^* \leq \infty$$

Def. (2.3.9 in GR by Rö):

1. $t_*$ or $t^*$ is called a physical singularity if $\rho \to \infty$ for $t \downarrow t_*$ or $t \uparrow t^*$.
2. $t_*$ is called a big bang if $f(t) \to 0$ and $f' \to \infty$ for $t \downarrow t_*$.
3. $t^*$ is called a collapse or big crunch if $f(t) \to 0$ and $f'(t) \to -\infty$ for $t \uparrow t^*$.

Example: If $\rho + 3p > 0$ and $H(t_0) > 0$ then M has a stationary singularity, i.e. $t_* \neq -\infty$ (not yet a physical singularity).

From (2) − (1):

$$0 \geq -\frac{4\pi}{3}(\rho + 3p) f = f''.$$

$$H(t_0) > 0 \Rightarrow f'(t_0) > 0 \Rightarrow f' \geq f'(t_0)$$

on $(t_*, t_0]$

$t_1 \in (t_*, t_0)$

$$f(t_0) > f(t_0) - f(t_1) = \int_{t_1}^{t_0} f'(t)\,dt \geq$$

$$\geq (t_0 - t_1) \cdot f'(t_0)$$

$$\Rightarrow \frac{1}{H(t_0)} = \frac{f(t_0)}{f'(t_0)} \geq (t_0 - t_1)$$

$$\Rightarrow \frac{1}{H(t_0)} \geq (t_0 - t_*) \Rightarrow t_* \neq -\infty$$

Prop. 2.3.11 (GR by Rör):

Prop. 2.3.11 (GR by Rov):

Suppose $t_*$ and $t^*$ are physical singularities if they are finite. Let $H(t_0) > 0$, $\rho > 0$ and suppose there are constants $-\frac{1}{3} < a < A$ s.t. $a \le \frac{p}{\rho} \le A$. Then

1. $t_*$ is a big bang.
2. If $\varepsilon = 0$ or $\varepsilon = -1$, then $t^* = \infty$ and $f \to \infty$, $\rho \to 0$ for $t \to \infty$
3. If $\varepsilon = 1$, then $t^* \neq \infty$ is a big crunch.

Example:

Let $p = 0$ (so called dust cosmos).

One has

$$0 = -f^2 8\pi p = f'^2 + \varepsilon + 2ff'$$

Solutions: $(c > 0)$

$\varepsilon = 0$: $f(t) = c \cdot (t - t_0)^{\frac{2}{3}}$

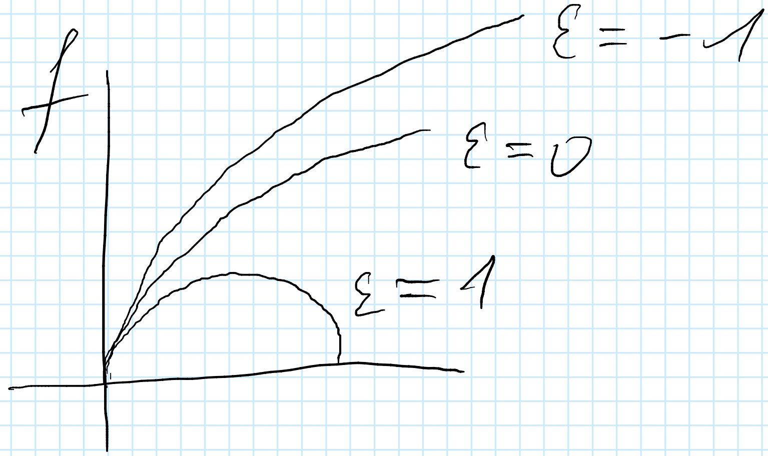
$\varepsilon = 1$:

$t(\alpha) := c(\alpha - \sin\alpha)$, $f(t(\alpha) + t_0) = c(1 - \cos\alpha)$

$t(\alpha) := C(\alpha - \sin\alpha),\ f(t(\alpha)+t_0) = C(1-\cos\alpha)$

$\varepsilon = -1$:

$t(\alpha) := C(\sinh\alpha - \alpha),\ f(t(\alpha)+t_0) = C(\cosh(\alpha)-1)$




Important: Spacetime itself shrinks.

Consider $(S, g_S) = (\mathbb{R}^3, g_{eucl})$,

$\gamma_1(s) = (s, 0),\ \gamma_2(s) = (s, x_0),\ x_0 \in \mathbb{R}^3\setminus\{0\}$

no spacial velocity component

$f_1, f_2$ constant $f_1 \neq f_2$

$c_{1,2}(s) = (s, \frac{s}{f_{1,2}^2} x_0)$ lightlike geodesics (photon)

$c_1$ meets $\gamma_1(0) = c_1(0) = 0 \in \mathbb{R}^{3,1}$

$\gamma_2(f_1^2) = c_1(f_1^2) = (f_1^2, x_0)$

$$\gamma_2(f_1^2) = c_1(f_1^2) = (f_1^2, x_0)$$

$c_2$ meets $\gamma_1(0) = c_2(0) = 0$

$$\gamma_2(0) = c_2(f_2^2)$$

$\Rightarrow$ flight duration has shrunk:

$$\Delta t_2 = f_2^2 < f_1^2 = \Delta t_1$$ but

light speed is still $c = 1$.