Prop 2.71 Let $A \subset M^m$ be achronal, $m = n+1$. TFAE

(i) $A \cap \text{edge}(A) = \emptyset$

(ii) A is a topological submfds.

Proof: (ii) $\Rightarrow$ (i):

Assume we have a chart $U \xrightarrow{\varphi} V$

as given by Lemma (3) 2.70a

for some some $p \in A = S$, $\varphi(p) = (q, 0)$

$I_\pm^U(p)$ open connected.

As A is achronal $I_+^U(p)$, A, $I_-^U(p)$

are pairwise disjoint.

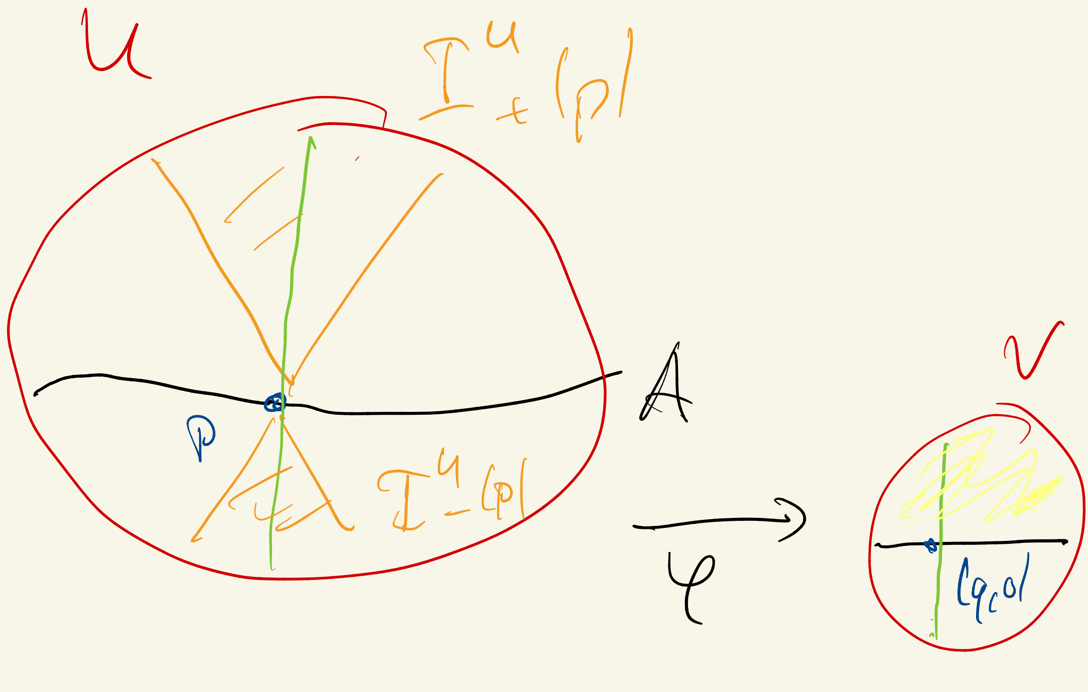
The timelike curve

$t \mapsto \varphi^{-1}(q, t)$ changes at $t=0$

from $I_-^U(p)$ to $I_+^U(p)$ and from

a connected comp. of $\varphi^{-1}(V \cap (\mathbb{R}^n \times \mathbb{R}_{<0}))$

to a connected comp of $\varphi^{-1}(V \cap (\mathbb{R}^n \times \mathbb{R}_{>0}))$



Thus $I_+^{\mathcal{U}}(p)$ and $I_-^{\mathcal{U}}(p)$

are in different conn. compon.

of $\mathcal{U}\setminus A$. Thus they cannot

be joined in $\mathcal{U}$ without

meeting A.

$\Rightarrow p \notin \mathrm{edge}(A)$

(ii) $\Rightarrow$ (iii):

let $p \in A$, $p \notin \mathrm{edge}(A)$.

Let $\tilde{\mathcal{U}}$ be an open nbhd of p s.th.

any f.d. timelike curve from

$I_-^{\tilde{\mathcal{U}}}(p)$ to $I_+^{\tilde{\mathcal{U}}}(p)$ meets A.

Wlog there is a diffeo

$$\begin{array}{ccccc} \hat{U} & \xrightarrow{\xi} & \xi(\hat{U}) \subset \mathbb{R}^m \cong \mathbb{R}^{n,1} \\ \cap & & \\ M & p \longmapsto 0 & \end{array}$$

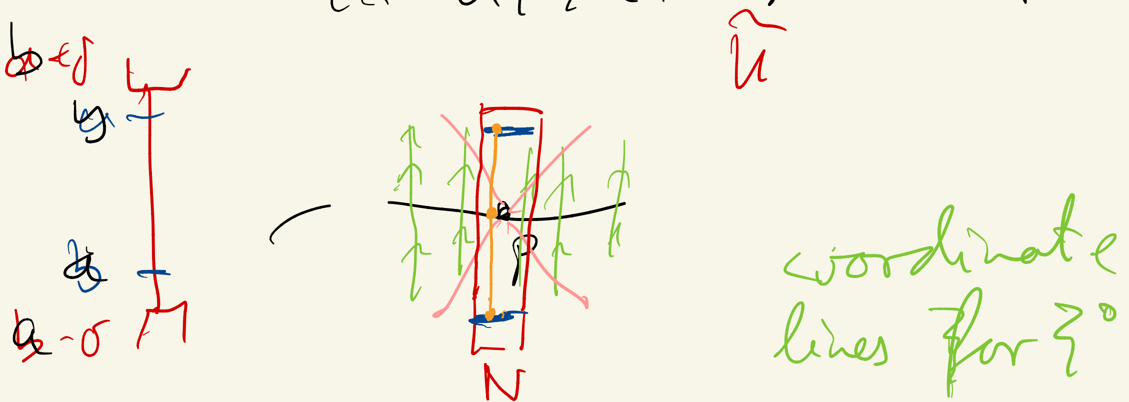
$\dfrac{\partial}{\partial \xi^0}$ is timelike f.d.

Then $\exists\, U \subset \hat{U}$ s.th. $p \in U$

1.) $\underset{\substack{\cap \\ \mathbb{R}^m}}{\xi(U)} = (a-\delta, b+\delta) \times N =: V$ $\quad a, b \in \mathbb{R}$, $a < b$, $\delta > 0$

2.) $\{x \in U \mid \xi^0(x) = b\} \subset I_+^{\hat{U}}(p)$

$\{x \in U \mid \xi^0(x) = a\} \subset I_-^{\hat{U}}(p)$



For $y \in N \subset \mathbb{R}^n$, the curve

$[a,b] \to U$, $\boxed{s \mapsto \xi^{-1}(s,y)}$

is timelike and meets A (by assumption) exactly once.

For $y \in A$ there is $s =: h(y)$ with $\xi^{-1}(s,y) \in A$.

Similarly as in previous lemma one shows that h is continuous. Proceed similarly as in the pf. of the lemma to get a topological hypersurface chart for A around p. $\square$

Corollary 2.72

Let $A \subset M^n$ be achronal.

TFAE (i) $\mathrm{edge}(A) = \emptyset$

(ii) A is a closed topological hypersurface.

Pf: "(i) $\Rightarrow$ (ii)" Prop 2.71 $\Rightarrow$ A top. hyp.

Since $\overline{A} \setminus A \subset \mathrm{edge}(A) = \emptyset$,

A is closed

"(ii) $\Rightarrow$ (i)": Prop 2.71 $\Rightarrow A \cap \mathrm{edge}(A) = \emptyset$

By defn $\mathrm{edge}(A) \subset \overline{A} = A \Rightarrow \mathrm{edge}(A) = \emptyset$

$\square$

Def 2.73 $B \subset M$ is a future set (past)

if $I_+(B) \subset B$

$I_-(B) \subset B$

Ex $J_+(0) \subset \mathbb{R}^{n,1}$

$\Vert$

$\{(x^0, \vec{x}) \in \mathbb{R}^{n,1} \mid x^0 \geq \|\vec{x}\|\}$ is a future set

B future set $\Longleftrightarrow$ $M \setminus B$ past set

Cor 2.76 Let $B \subset M$ be a future set. Then ∂B is achronal a closed top hypersurface.

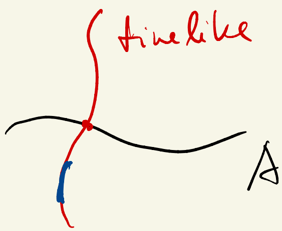
Pf: Script of Bär.

Summary: Cauchy hypersurfaces and globally hyperbolic manifolds

$A \subset M$ achronal

$\iff$ no timelike f.d. curve from A to A

$\iff$ every timelike f.d. curve hits A at most once.



Idea: A Cauchy hypersurface is a subset which hit precisely once for every "inextendible" timelike f.d. curve.

Def.: A causal f.d. piecew. C^1-curve $c: I \to M$, I open, is C^0-future (past) inextendible if

$\lim_{t \nearrow \sup I} c(t)$ ($t \searrow \inf I$) does not exist.

(Coinciding with the def. in Ex. sheet 6 no. 4, due Ex. no 4 a)).

c is called inextendible (C^0-inextendible) if it is past and future inextendible.

<u>Def 2.77</u> A Cauchy hypersurface in a time-orient. mfd M is a subset S, such that any inextendible (f.-d.) timelike curve hits S exactly once.

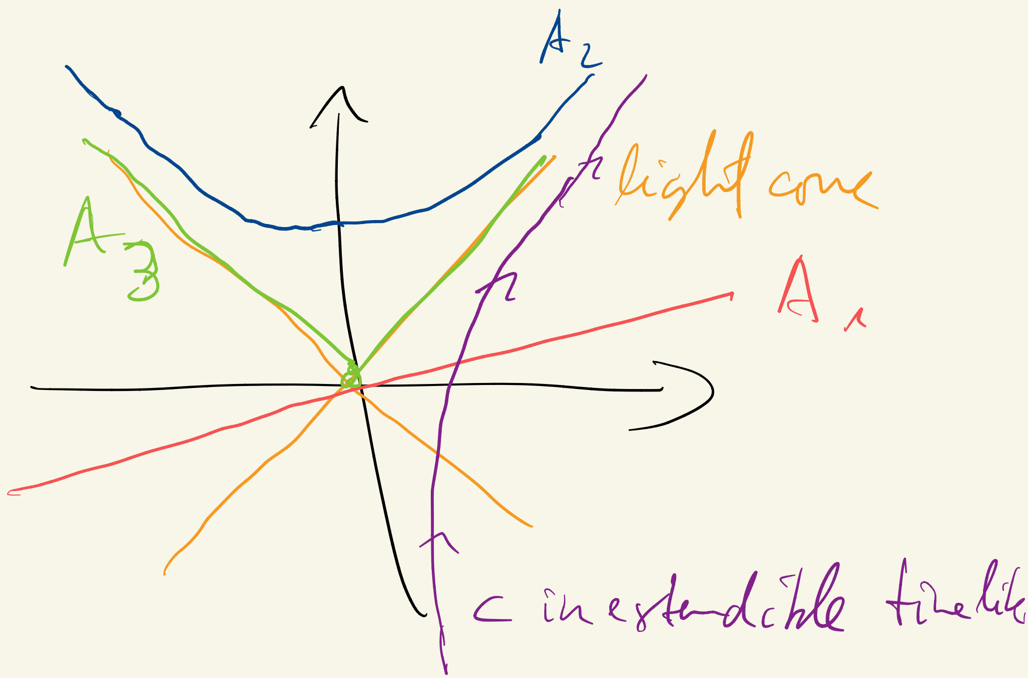
Example 2.78

$A_i \subset \mathbb{R}^{1,1}$ achronal

$A_1 = \{$ spacelike line in $\mathbb{R}^{1,1}\}$

$A_2 = \{(\sqrt{x^2+1}, x) \mid x \in \mathbb{R}\} = H^1$

$A_3 = \{(|x|, x) \mid x \in \mathbb{R}\} = C_+(0)$



A_1 Cauchy hyp.

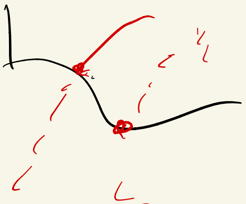
A_2, A_3 are not.

Prop 2.80 If $S \subset M$ is a Cauchy hypersurface. Then

(i) S is achronal

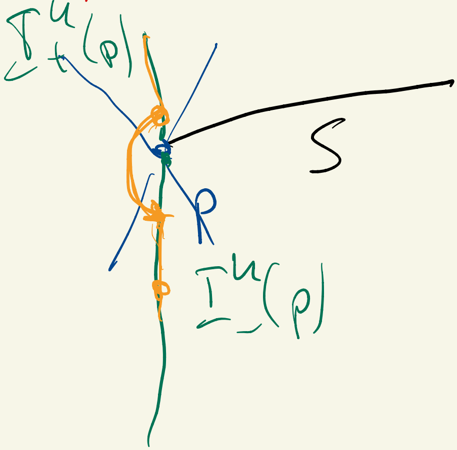
(ii) S is a closed topological hypersurface

(iii) Every inextendible causal curve hits S.

Pf: (i)  A

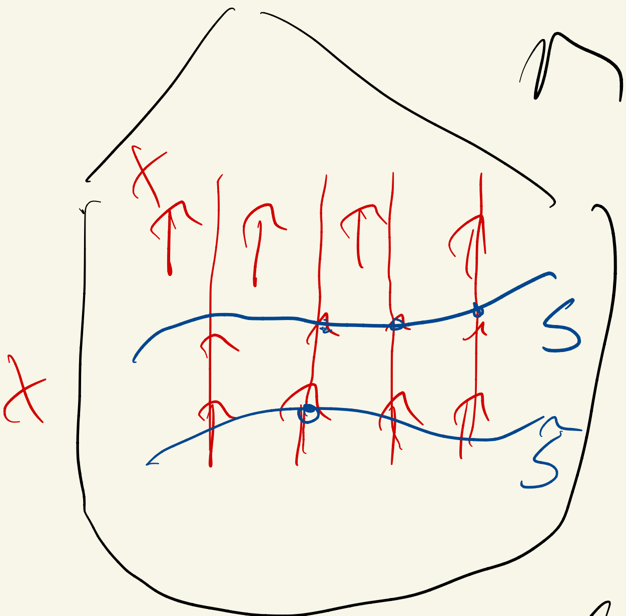
Every smooth/p.w. C^1- ~~cu~~ ~~causal~~ timelike curve $c:[a,b] \to M$ can be extended to an inextendible one

(ii) S is not a closed topological hyp.,
then $\mathrm{edge}(S) \neq \emptyset$



(iii) see script of Bär

<u>Note</u> Any time-oriented Lorentzian mfd M carries a timelike (f.-d.) vector field $X \in \Gamma(TM)$



Let $\varphi_\bullet : \mathcal{U} \to M$, $(s,x) \mapsto \varphi_s(x)$ be the flow of X.

$\mathcal{U} \subset \mathbb{R} \times M$

$\cup$

$\{0\} \times M$

$$\varphi_\bullet(s,x) = \varphi_s(x)$$

flow $\overset{\text{def}}{\iff}$ $\varphi_0 = id_X$

$$\frac{\partial}{\partial s}\varphi(s,x) = X|_{\varphi(s,x)}$$

$s \mapsto \varphi(s,x)$ inextendi timelike curve

$x \in M$: $I_x := \{s \in \mathbb{R} \mid (s,x) \in \mathcal{U}\}$ intervalle

$\mathcal{U}$ maximal

Let S be a Cauchy hypersurface in M. Let $\sigma^S(x)$ be the unique $s \in I_x$ s.th.

$\varphi(s,x) \in S$, σ^S conti.

$\rho^S : M \longrightarrow S$ homotopy equiv.

$\rho(x) := \varphi(\sigma^S(x), x)$ $\quad \rho^S|_S = S$

$\hat{S}$ is another Cauchy hypers.

$$\hat{S} \xleftarrow{\rho^{\hat{S}}} M \xrightarrow{\rho^S} S$$

$$x \longmapsto \varphi(\sigma^{\hat{S}}(x), x)$$

$\rho^S|_{\hat{S}}$ bijecti

$\rho^{\hat{S}}|_S$

$\Rightarrow \mathcal{S}^{S}|_{\hat{S}}$ is a homeom.

(with $\mathcal{S}^{\hat{S}}|_{S}$).

Def. 2.85

A subset $\Omega \subset M$ is *globally hyperbolic*
open
if

1.) the (strong) causality holds on Ω
(Def. 2.19)

2.) For all $p, q \in \Omega$ the causal diamonds

$J(p,q) := J_+(p) \cap J_-(q)$ are
compact and contained in Ω.

You may remove "strong" without changing the defn.
(Theorem 2006)

$M = \Omega$

TFAE

1) M is globally hyperbolic

2) M has a Cauchy hypersurface

3) M has a Cauchy hypersurface that is a smooth spacelike hypersurface.

All cond. imply $M \underset{\text{diffeom}}{\cong} S \times \mathbb{R}$

Cauchy hypers.