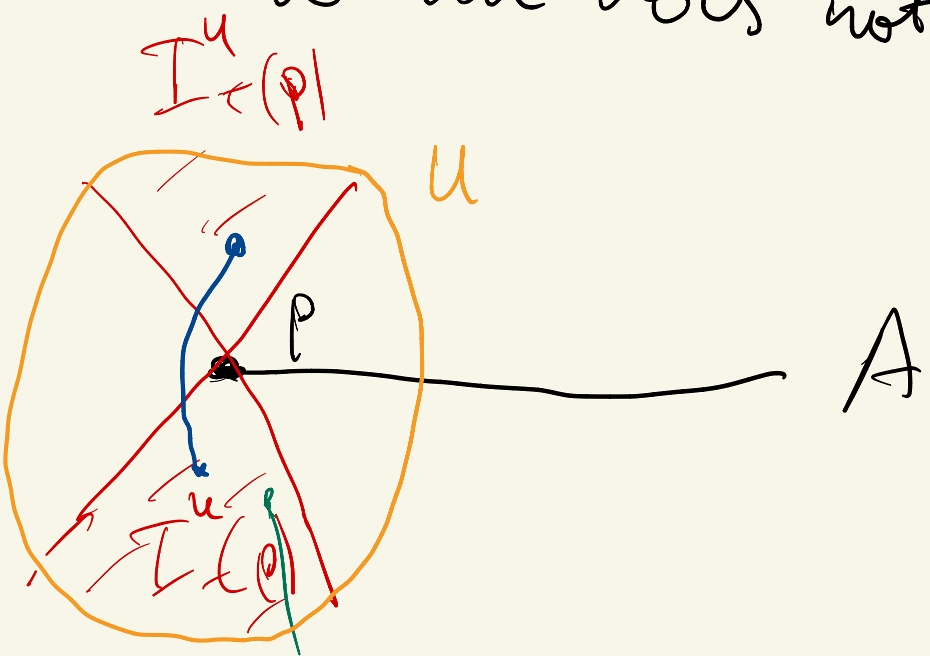
Def 2.65

The *edge* of an achronal subset A is defined as

$$\mathrm{edge}(A) := \left\{ p \in \overline{A} \;\middle|\; \begin{array}{l} \text{for all open nbhds } U \text{ of } p \text{ there is a} \\ \text{timelike (future-directed) piecewise } C^1\text{-curve in } U \\ \text{from } I_-^U(p) \text{ to } I_+^U(p) \\ \text{which does not hit } A. \end{array} \right\}$$

$\mathbb{R}^{1,1}$

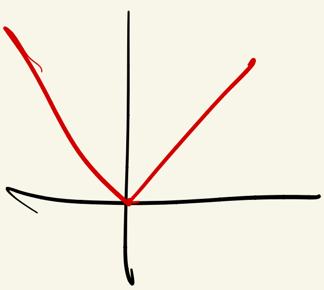


There are no such past-directed curves as this would contradict achronality, so we could replace future directed, by future- or past-directed curves.

<u>Examples 2.66</u> $M=\mathbb{R}^{n,1}$

1.) a) $A=\left\{\begin{pmatrix}t\\x\end{pmatrix} \;\middle|\; t=|x|\right\} =: C_+(0)$

future lightcone

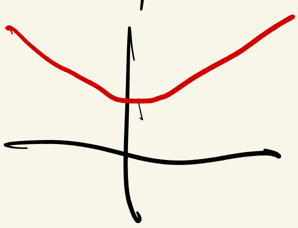


achronal, not acausal

$\mathrm{edge}(A)=\emptyset$.

1.) b) $A=\left\{\begin{pmatrix}t\\x\end{pmatrix} \;\middle|\; t^2=|x|^2+1\right\}$

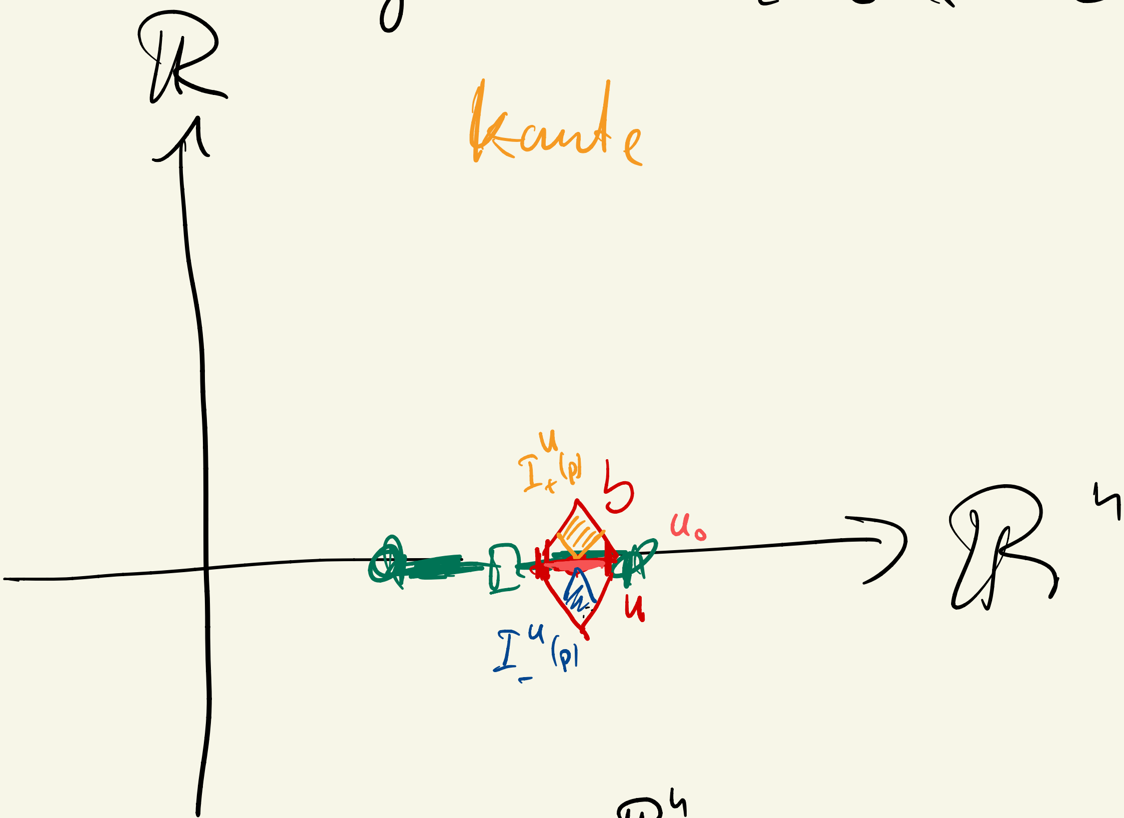
acausal



$\mathrm{edge}(A)=\emptyset$

2) $B \subset \mathbb{R}^n$, $A := \{0\} \times B \subset \mathbb{R}^{n+1}$

then $\mathrm{edge}(A) = \{0\} \times \partial B$.



If $b \in \mathring{B}$, say $B_\varepsilon^{\mathbb{R}^n}(b) =: U_0 \subset \mathring{B}$, $\varepsilon > 0$

$$U = \left\{ \begin{pmatrix} x^0 \\ x \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; |x^0| + \|x - b\| < \varepsilon \right\}$$

open nbhd of $\{(0,b)\}$. $\qquad p = (0,b)$

$$I_\pm^U((0,b)) = \left\{ \begin{pmatrix} x^0 \\ x \end{pmatrix} \in U \;\middle|\; x^0 > 0 \ (x^0 < 0), \ \|x - b\| > |x^0| \right\}$$

Any timelike, f.-d. curve from

$I^U_-(p)$ to $I^U_+(p)$ in U has to

meet $\underset{\{0\}\times\overline{B}}{U_0} \subset A$. $\Rightarrow$ $p=(0,b) \in$ edge.

$p \in \overline{A} \Rightarrow p \in$ edge by definition

$p = \begin{pmatrix} x^0 \\ x \end{pmatrix}$, $x^0 \neq 0 \Rightarrow p \notin \mathrm{edge}(A)$

$p = \begin{pmatrix} 0 \\ b \end{pmatrix}$, $b \in \partial B$.

For every nbhd V of b in $\mathbb{R}^2$ we have $V \setminus B \neq \emptyset$

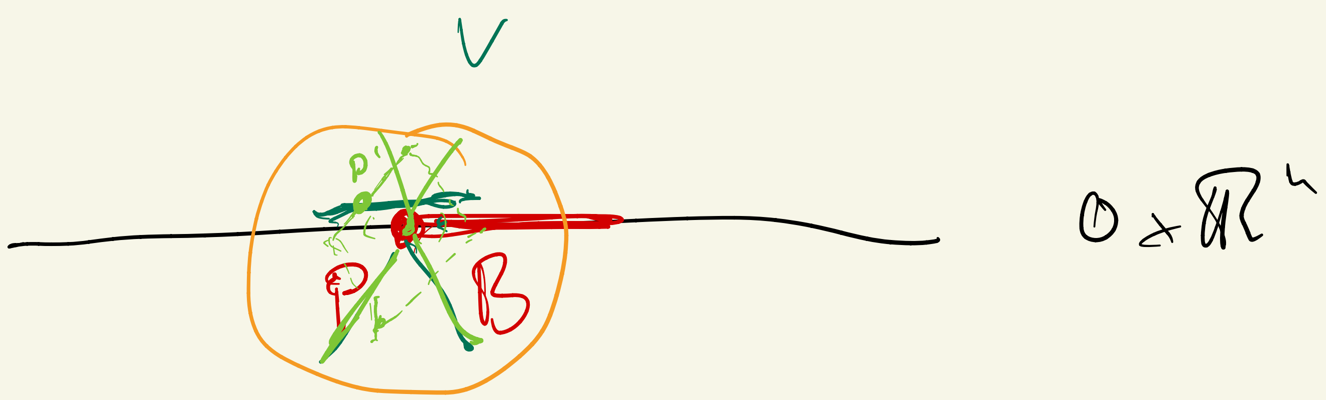
$\underset{b'}{\underset{\Downarrow}{}}$ $\quad p' = (0, b')$

For any U (nbhd of p in $\mathbb{R}^{n+1}$)
$\hat{\ }$
open

choose V sufficiently small s.th. $0 \times V \subset U$.

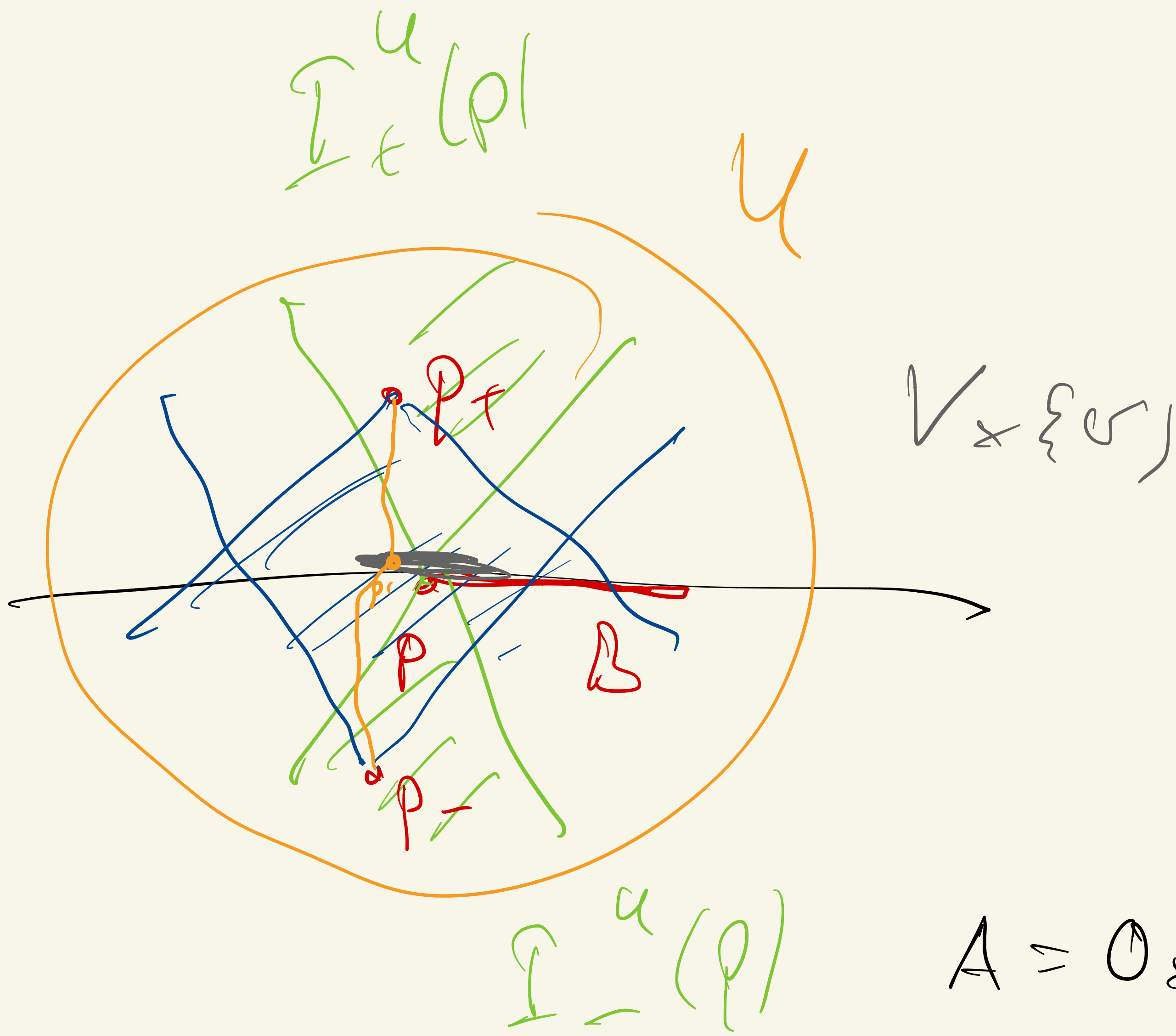
$b' \in V \setminus B$ $\quad p' = \begin{pmatrix} 0 \\ b' \end{pmatrix} \in \mathbb{R}^{n+1}$

Choose $p_- \in I_-^u(p)$, $p_+ \in I_+^u(p)$



$I_+^u(p_-) \cap I_-^u(p_+)$ open nbhd of p. Wlog $0 \times V \subset I_+^u(p_-) \cap I_-^u(p_+)$

Choose a timelike path from p_- to p' and compose it with a timelike path from p' to p_+



$A = O \cap B$

$\Rightarrow \quad p \in \mathrm{edge}(A)$

Remark 2.67

If A is achronal, then

$\bar{A} \setminus A \subset \text{edge}(A)$.

Let $p \in \bar{A} \setminus A$. $\mathcal{U}$ is nbhd of

p. Then there

exist a f.d. timelike curve from

$I_-^{\mathcal{U}}(p)$ to p, and a f.d.

timelike curve from p

to $I_+^{\mathcal{U}}(p)$. Composing them yields a curve as required in the defn of edge (A).

$\Rightarrow$ $p \in \mathrm{edge}(A)$.

Lemma 2.68 If A is achronal, then edge (A) is closed.

Pf: later or script Bär

Def 2.69 Topological hypersurface

If M carries a C^k-atlas,
it is also a C^l-atlas,
for $0 \leq l \leq k$

In particular every C^∞-mfld
is a topological mfld (C^0-mfld)

In a C^k-mfld M a C^l-submfld
is a subset of M for which
a C^l-submfld chart exists.
$(l \leq k)$

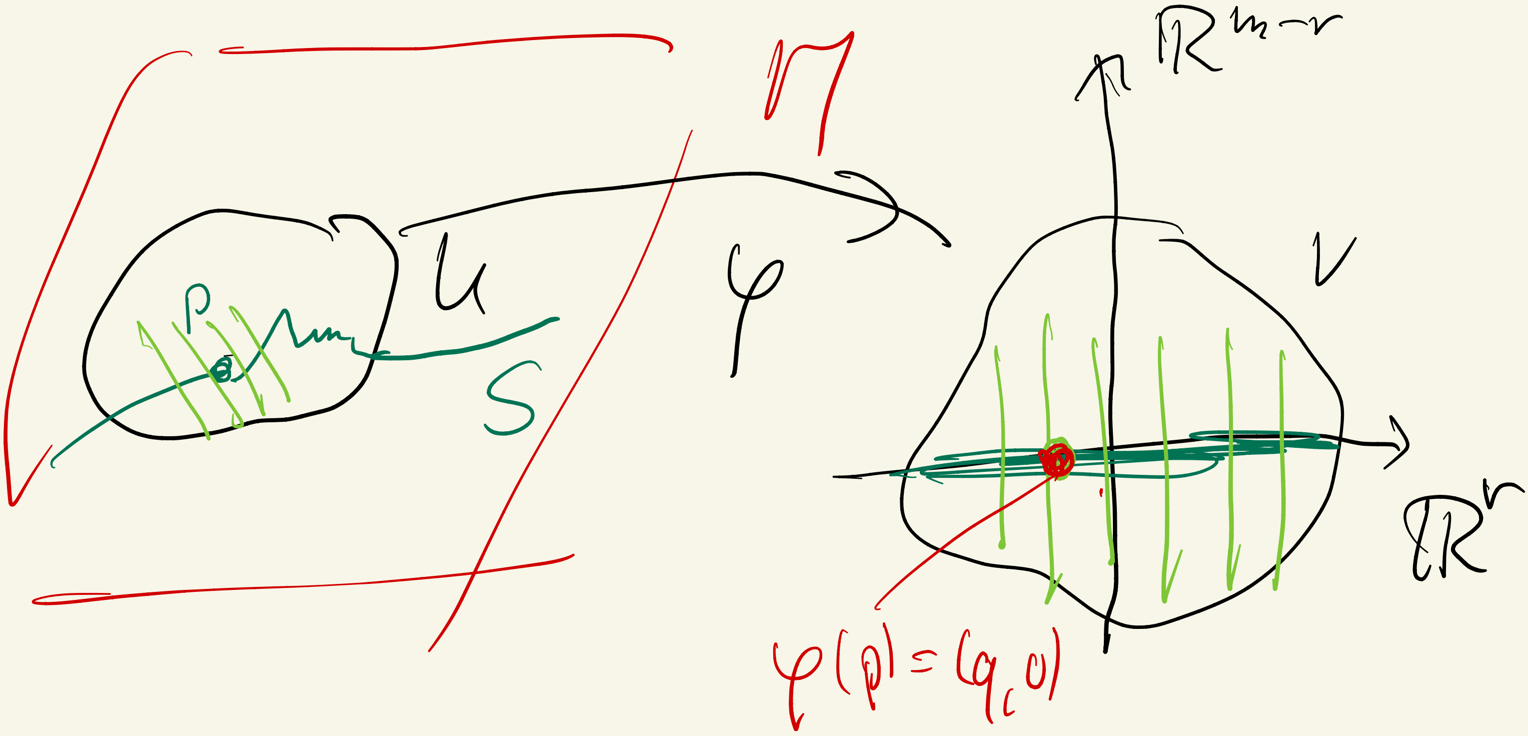
In particular a topological (C^0-) submfd of dim r of a C^k-mfd M^m is a subset $S \subset M$ s.th. for all $p \in S$ there is a

$U \subset_o M$ and a homeom.
$U \ni p$

$\varphi : U \to V \subset_o \mathbb{R}^m$ with

$$\varphi(U \cap S) = V \cap (\mathbb{R}^r \times \underbrace{\{0\}}_{\in \mathbb{R}^{m-r}})$$

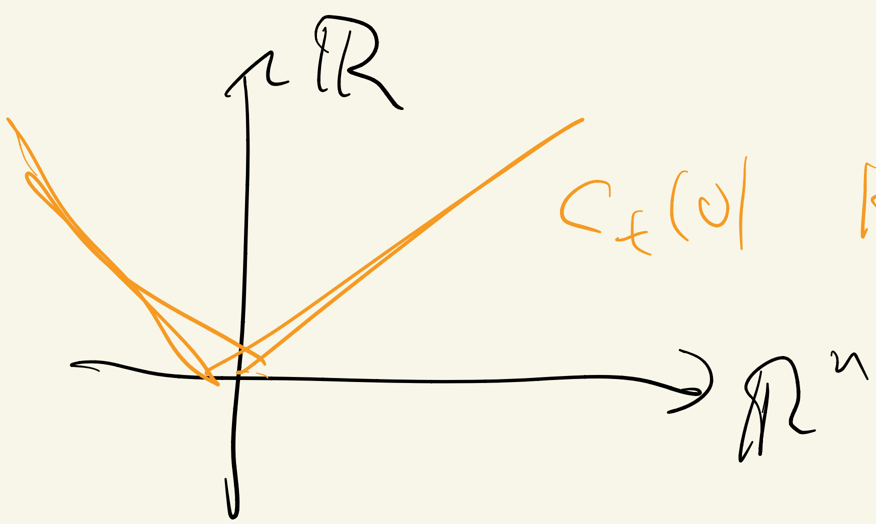
If $r = m-1$, then M is a top. hypersurf.



Example 2.70

$$S := C_+(0) \subset \mathbb{R}^{n,1}$$ top. hypersurf.

$$= \left\{ \begin{pmatrix} \|\vec{x}\| \\ \vec{x} \end{pmatrix} \;\middle|\; \vec{x} \in \mathbb{R}^n \right\}$$



future part of lightcone $\cup \{0\}$

Lemma 2.70a (Repairs a gap in O'Neill and in Bär's script?)

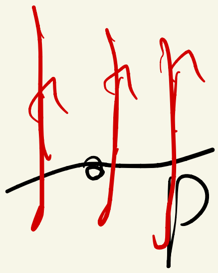
Let $S \subset M^m$, $m = n+1$, be achronal and a topological hypersurface, then for any $p \in S$ we can choose the homeom. φ as in Def 2.69 with the additional property that $t \mapsto \varphi^{-1}(x, t)$ is a smooth timelike curve.

Proof: Assume $p \in S$, U, V, φ as in Def 2.69. $p \in U$.

$\varphi(p) = (q, 0)$

1.) If $U_0 \subset U$ is a nbhd of p in M, then $U_0 \setminus S$ is not connected.

2.) Let N^n be a spacelike smooth hypersurface through p $(p \in N)$

 N $\quad T \in \Gamma(TM|_N)$

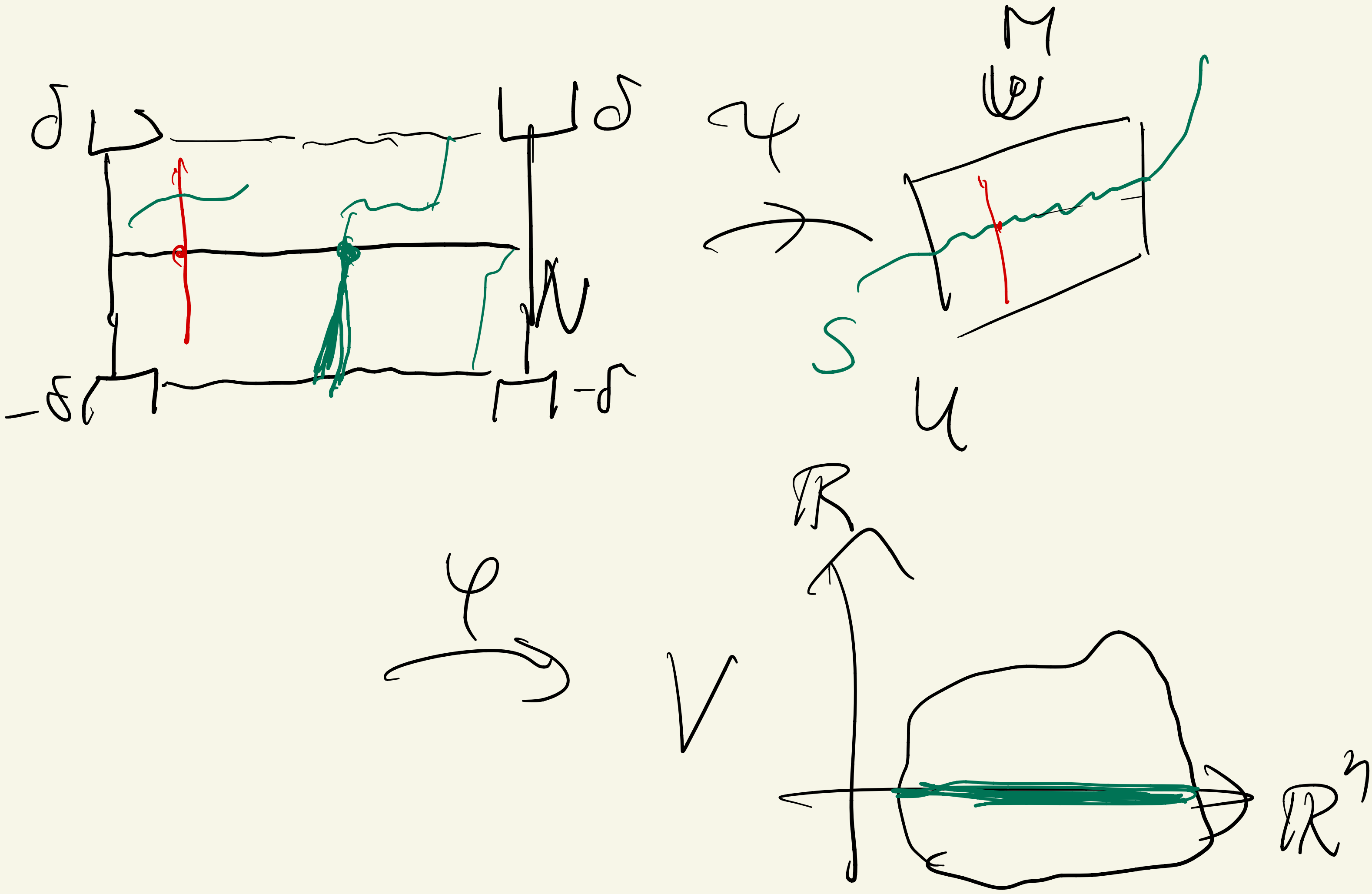
T timelike

By choosing N and $\delta > 0$ small we may achieve that

$$N \times (-\delta, \delta) \xrightarrow{\Psi} M$$
$$(x, t) \longmapsto \exp_x(t\,T(x))$$

is a diffeo onto its image, and image $\Psi \subset U$

wlog - (possibly shrinking U) we may assume image $\Psi = U$.



3.) Achronality of S implies that for any $x \in S$ there is at most one $\tau(x) \in (-\delta, \delta)$ s.th.

$$\psi(x, \tau(x)) = \exp_x(\tau(x) T(x)) \in S$$

$$D := \{x \in N \mid \tau(x) \text{ as above exists}\}$$

$\Psi^{-1}(\varphi^{-1}(V_\alpha \cap (\mathbb{R}^m \times \{0\}))))$

$= \Psi^{-1}(U \cap S) = \text{graph}\{\tau : D \to (-\delta, \delta)\}$

A graph of a fctn is closed in $D \times (-\delta, \delta) \Longleftrightarrow \tau$ is continuous.

$U \cap S$ is closed in U

$\Rightarrow$ graph $(\tau : D \to (-\delta, \delta))$ is closed in $N \times (-\delta, \delta)$

$\Rightarrow \tau$ is continuous

4.) $\mathcal{D} \ni p$ is open

$$V \cap (\mathbb{R}^n \times \{0\}) \xrightarrow[\cong]{\psi^{-1} \circ \varphi^{-1}} \mathrm{graph}(T)$$

$$\xrightarrow[\text{proj. to } N]{\pi_1} \mathcal{D} \hookrightarrow N^n$$

injective continuous

<u>Theorem</u> (invariance of domain)

If $F : U \to \mathbb{R}^n$ is continuous and injective, $U \overset{\circ}{\subset} \mathbb{R}^n$, then $F(U)$ is open in $\mathbb{R}^n$ and F is a homeom.

<u>Conclusion</u>: $\mathcal{D}$ is open in N.

5.) Thus by restricting N further we may assume $D = N$.

By restricting N further, using continuity of τ we may assume $\tau(N) \subset \left(-\frac{\delta}{2}, \frac{\delta}{2}\right)$

wlog $\exists\, g : \underset{\substack{\text{open ball} \\ \text{in } \mathbb{R}^4}}{W_1} \xrightarrow[\text{diffeo}]{} N$ ($\rightsquigarrow$ chart of N)

So far

$\psi : N \times (-\delta, \delta) \to M$

smooth diffeom. onto its image U_1 which is an open nbhd of p.

$\Rightarrow U_1 \xrightarrow{\psi^{-1}} N \times (-\delta, \delta) \xrightarrow{(g \times id)^{-1}} W_1 \times (-\delta, \delta)$

$\psi^{-1}(U_1 \cap S) = \text{graph}(\tau)$

continuous

$$\hat{\Psi} : N \times \left(-\frac{\delta}{2}, \frac{\delta}{2}\right) \longrightarrow M$$

$$(x,t) \longmapsto \Psi(x, t + \tau(x))$$

homeom. onto its image

$U_2 \subset_o U_1$, $p \in U_2$.

$\hat{\Psi}(p,0) = p$

$\hat{\Psi}^{-1}(U_2 \cap S) = N$

Thus $\varphi_2 := (g \times id)^{-1} \circ \hat{\Psi}^{-1}$:

$$U_2 \longrightarrow W_1 \times \left(-\frac{\delta}{2}, \frac{\delta}{2}\right) =: V_2$$

$U_2 \xrightarrow{\varphi_2} V_2$ is a topolog. submfd chart for S around

p, s.th.

$t \longmapsto \varphi_2^{-1}(x, t)$

$\cap$

$(-\frac{\delta}{2}, \frac{\delta}{2})$

is smooth & timelike

$\square$