Lemma 2.43

Let $P \subset M$ be a spacelike submfd of a Lorentzian mfd, and $c : [0, b] \to M$ a lightlike geodesic with $c(0) =: p \in P$, but $\dot{c}(0) \notin N_p P$. Then in every nbhd of c we find a timelike curve from P to $q := c(b)$.

Proof: There is $X_0 \in T_p P$ with $\langle X_0, \dot{c}(0) \rangle > 0$.

Define $X \in \Gamma(c^* TM)$ by $X(0) = X_0$, $\frac{\nabla}{dt} X = 0$.

Define $V(t) := \left(1 - \frac{t}{b}\right) X(t) \Rightarrow V(0) = X_0, V(b) = 0$

Choose a variation c_s of c with $\frac{\partial c_s}{\partial s}\big|_{s=0} = V$, $c_s(0) \in P$ and $c_s(b) = q$ for all s.

c lightlike $\Rightarrow \langle \dot{c}_s, \dot{c}_s \rangle \big|_{s=0} = 0$

$$\frac{\partial}{\partial s}\Big|_{s=0} \langle \dot{c}_s, \dot{c}_s \rangle = 2 \left\langle \frac{\nabla}{ds} \frac{\partial c_s}{\partial t}, \dot{c}_s \right\rangle \Big|_{s=0} = 2 \left\langle \frac{\nabla}{dt} V, \dot{c} \right\rangle$$

$$= -\frac{2}{b} \langle X, \dot{c} \rangle = -\frac{2}{b} \langle X(0), \dot{c}(0) \rangle$$

$$< 0 .$$

$\Rightarrow \exists s_0 > 0 \ \forall s \in (0, s_0)$

c_s is timelike. $\square$

Theorem 2.44

Let M be a Lorentzian mfd, $P \subset M$ spacelike submfd, and let $c : [0,b] \to M$ be a causal curve from $c(0) =: p \in P$ to $q := c(b)$. Then in every (compact-open) neighborhood of c there is a timelike curve from P to q, unless c is, up to parametrization, a lightlike geodesic with $\dot{c}(0) \in N_p P$ such that P has no focal values $t_0 < b$ along c.

Proof: If c is not a lightlike (pre-) geodesic, we find a timelike curve from p to q in a nbhd of c.

Thus wlog c is a lightlike geodesic. Use Lemma 2.43 to see that the theorem holds for $\dot{c}(0) \not\perp T_p P$.

Prop 2.38 gives the statement if there no focal points values in $(0,b)$ in the case $\dot{c}(0) \perp T_p P$.

<u>Recall</u> X ~~timelike~~ $\in TM$ $|X| := \sqrt{|g(X,X)|}$
$= \sqrt{|\langle X, X\rangle|}$

For causal curves $c : [a,b] \to M$

$$\mathcal{L}[c] = \int_a^b |\dot{c}(t)| \, dt$$

proper time

c parametrized by propertime

$$\Longleftrightarrow \; g(\dot{c}(t), \dot{c}(t)) \equiv -1$$

Prop 2.44 a (First variation formula for timelike curves in Lorentzian mflds)

M a Lorentzian mfd, $c: [a,b] \to M$ a timelike smooth, parametrized by proper time. Let $c_\cdot : [a,b] \times (-\varepsilon, \varepsilon) \to M$, $(t,s) \mapsto c_s(t)$ a smooth variation of c, $V(t) := \frac{\partial c_s}{\partial s}(t,0)$.

Then
$$\frac{\partial}{\partial s}\Big|_{s=0} \mathcal{L}[c_s] = \int_a^b \langle V, \frac{\nabla}{dt} \dot{c} \rangle \, dt$$
$$+ \langle V(a), \dot{c}(a) \rangle - \langle V(b), \dot{c}(b) \rangle .$$

Note: Lorentzian analogue of Thm 3.2.8 of Clara Löh.

Proof $\frac{d}{ds} L[c_s] = \int_a^b \frac{\partial}{\partial s} \sqrt{-\langle \dot{c}_s, \dot{c}_s \rangle}\, dt$

$$= \int_a^b \frac{1}{2} \frac{-2 \langle \frac{\nabla}{ds} \dot{c}_s, \dot{c}_s \rangle}{\sqrt{-\langle \dot{c}_s, \dot{c}_s \rangle}}\, dt$$

$$= -\int_a^b \langle \frac{\nabla}{dt} \frac{\partial c_s}{\partial s}, \frac{\dot{c}_s}{|\dot{c}_s|} \rangle\, dt \overset{s=0}{=}$$

$$= -\int_a^b \langle \frac{\nabla}{dt} V, \dot{c} \rangle\, dt$$

$$= \underbrace{-\int_a^b \frac{d}{dt} \langle V, \dot{c} \rangle\, dt}_{\langle V(a), \dot{c}(a) \rangle - \langle V(b), \dot{c}(b) \rangle} + \int_a^b \langle V, \frac{\nabla}{dt} \dot{c} \rangle\, dt$$

$\square$

In particular $p, q \in M$

$\tilde{c}$ is stationary point of

$$\left\{ c \text{ timelike } C^2\text{-curves} \right.$$

parametr. by proper time

$$\left. \text{from } p \text{ to } q \right\} \xrightarrow{\mathcal{L}} \mathbb{R}$$

$\Longleftrightarrow$ $\tilde{c}$ is a ~~pre~~-geodesic.

Prop 2.113 (Second var. formula)

Let M, c, c_0, V as above,

$$A(s) = \frac{\nabla}{ds}\Big|_{s=0} \frac{\partial c_s}{\partial s} \in \Gamma(c^* TM).$$

Additionally we assume $c = c_0$ is (timelike) geodesic.

Then

$$\frac{\partial^2}{\partial s^2}\Big|_{s=0} \mathcal{L}[c_s] = \langle A(a), \dot{c}(a) \rangle - \langle A(b), \dot{c}(b) \rangle - \int_a^b \langle R(V, \dot{c}) V, \dot{c} \rangle \, dt$$

$$- \int_a^b \langle \pi^{nor}\left(\frac{DV}{dt}\right), \pi^{nor}\left(\frac{DV}{dt}\right) \rangle \, dt$$

Here
$$\frac{DV}{dt} = \overbrace{\pi^{tan}\left(\frac{DV}{dt}\right)}^{\in \mathbb{R}\dot{c}} + \overbrace{\pi^{nor}\left(\frac{DV}{dt}\right)}^{\perp \dot{c}}$$

$$\pi^{tan}\left(\frac{DV}{dt}\right) = \frac{D}{dt}\pi^{tan}(V)$$

$$\pi^{nor}\left(\frac{DV}{dt}\right) \neq \frac{D}{dt}\pi^{nor}(V)$$

$$\frac{D}{dt}\pi^{tan}(V) = \frac{D}{dt}\langle V, \dot{c}\rangle \dot{c}$$

<u>Proof:</u> $\displaystyle \frac{\partial^2}{\partial s^2}\Big|_{s=0} \mathcal{L}[c_s] =$

$$- \int_a^b \left\langle \frac{D}{ds}\Big|_{s=0} \frac{D}{dt}\frac{\partial c_s}{\partial s}, \dot{c} \right\rangle dt$$

$$- \int_a^b \left\langle \frac{D}{dt}\frac{\partial c_s}{\partial s}\Big|_{s=0}, \frac{D}{ds}\Big|_{s=0}\left(\frac{\dot{c}_s}{\|\dot{c}_s\|}\right) \right\rangle dt$$

$$= -\int_a^b \langle R(V, \dot{c})V, \dot{c}\rangle\, dt - \int_a^b \langle \frac{D}{dt} \overbrace{\frac{D}{ds} \frac{\partial c_s}{\partial s}\Big|_{s=0}}^{A}, \dot{c}\rangle\, dt$$

$$- \int_a^b \langle \frac{D}{dt} V(t), \pi^{hor}\left(\frac{DV}{dt}\right)\rangle\, dt = rhs$$

as $\int_a^b \langle \frac{D}{dt} A, \dot{c}\rangle\, dt = \int_a^b \frac{d}{dt} \langle A, \dot{c}\rangle\, dt$

$$= \langle A(b), \dot{c}(b)\rangle - \langle A(a), \dot{c}(a)\rangle$$

$$\frac{D}{ds}\Big|_{s=0} \left(\frac{\dot{c}_s}{|\dot{c}_s|}\right) = \frac{D}{ds}\Big|_{s=0} \dot{c}_s - \frac{1}{2} \frac{(-2) g\left(\frac{D}{ds}\dot{c}_s, \dot{c}\right)}{\sqrt{-g(\dot{c}_0, \dot{c}_0)}^{3}} \dot{c}_0$$

$$= \underbrace{\frac{D}{dt} \frac{\partial c_s}{\partial s}\Big|_{s=0}}_{\frac{D}{dt} V} + \frac{g\left(\frac{D}{dt} \frac{\partial s}{\partial s}\Big|_{s=0}, \dot{c}\right)}{1} \dot{c}$$

$$\frac{D}{dt} V \;\Big|\; = \frac{D}{dt} V + \langle \frac{D}{dt} V, \dot{c}\rangle \dot{c} = \pi^{hor}\left(\frac{DV}{dt}\right)$$

3.2.5 Cauchy hypersurfaces

M a connected, time-oriented Lorentzian mfd.

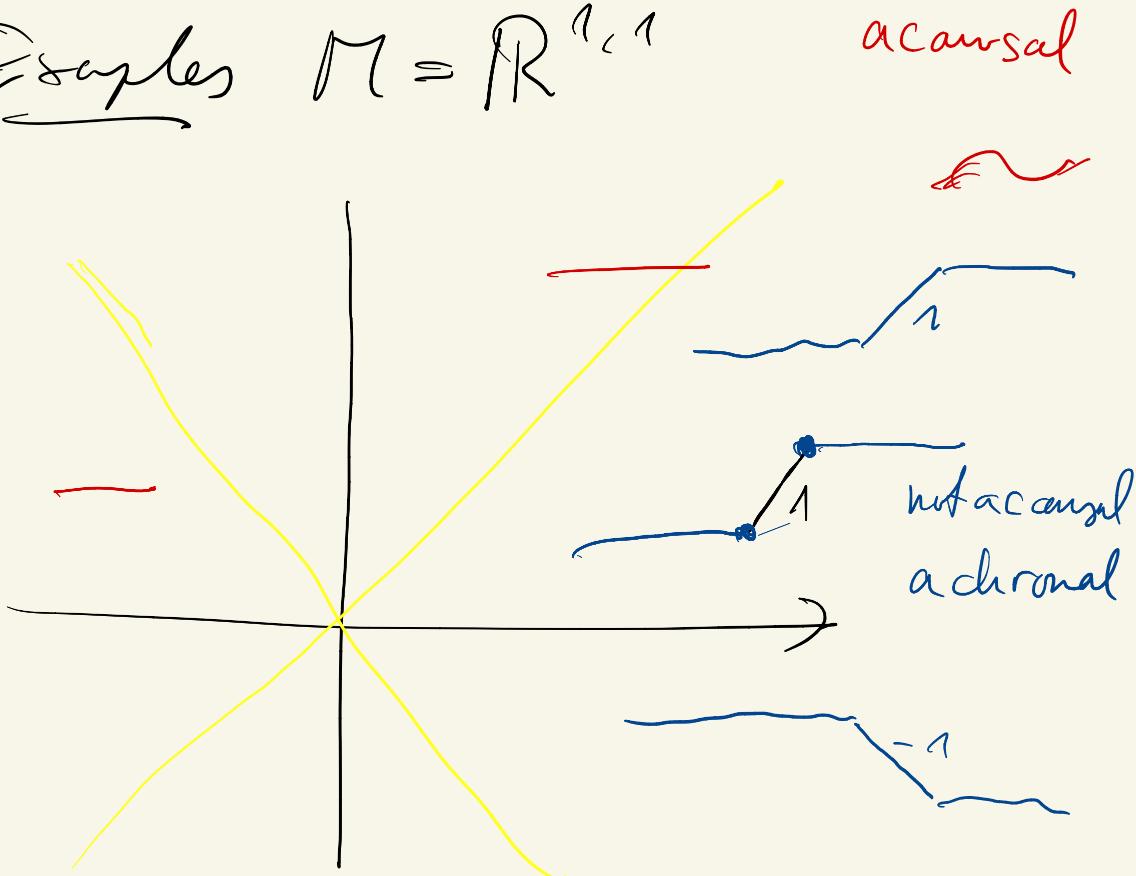
<u>Def</u> 2.62

$A \subset M$ is <u>achronal</u> if there are no $p, q \in A$ with $p \ll q$ (i.e. no timelike curve hits A at least twice)

$A \subset M$ is <u>acausal</u> if there are no $p, q \in A$ with $p < q$ (i.e. no causal curve hits A at least twice) acausal $\Rightarrow$ achronal

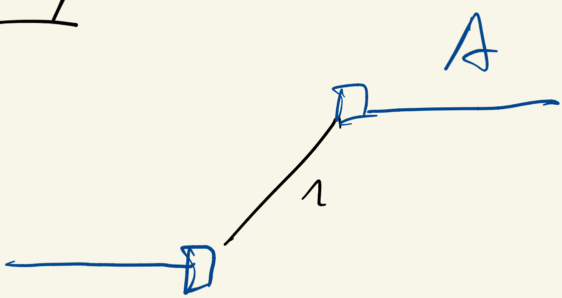
The Def 2.62 is the same if we use smooth curves or piecewise C^1-curves.

<u>Examples</u> $M = \mathbb{R}^{1,1}$



The closure of an achronal set is achronal: Suppose $p, q \in \overline{A}$ with $p \ll q$. $\exists\, p_n, q_n \in A$ $p_n \to p$, $q_n \to q$. Prop 2.14 implies that for $\forall n \geq n_0$ we have $p_n \ll q_n \Rightarrow A$ not achronal.

Example



A acausal

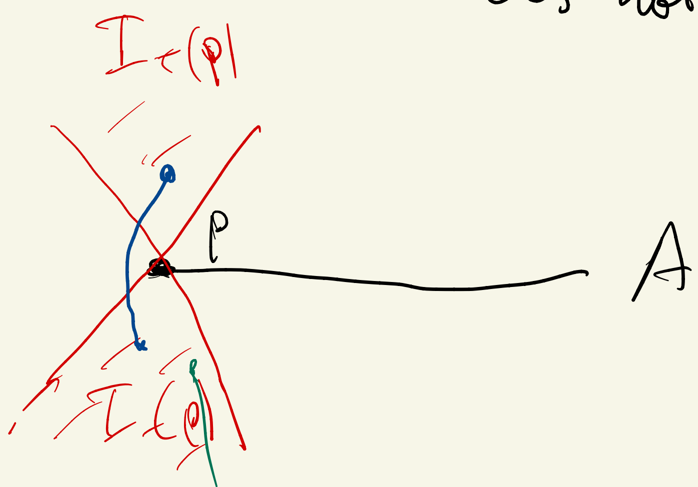
$\overline{A}$ not acausal

Def 2.65

The *edge* of an achronal subset A is defined as

$$\mathrm{edge}(A) := \left\{ p \in \overline{A} \;\middle|\; \begin{array}{l} \text{for all open nbhds } U \text{ of } p \text{ there is a} \\ \text{timelike (future-directed) piecewise } C^1\text{-curve in } U \\ \text{from } I_-^U(p) \text{ to } I_+^U(p) \\ \text{which does not hit } A. \end{array} \right\}$$

$\mathbb{R}^{1,1}$

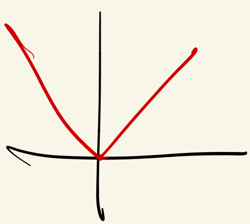


There are no such past-directed curves as this would contradict achronality, so we could replace future directed by future- or past-directed curves.

Examples 2.66 $M=\mathbb{R}^{n,1}$

1.) a) $A=\left\{\begin{pmatrix} t \\ x \end{pmatrix} \,\middle|\, t=|x|\right\} =: C_t(0)$

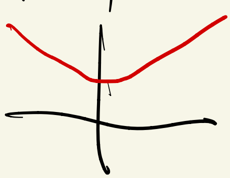
future lightcone



achronal, not acausal

$\mathrm{edge}(A)=\emptyset$.

1.) b) $A=\left\{\begin{pmatrix} t \\ x \end{pmatrix} \,\middle|\, t^2=|x|^2+1\right\}$



acausal

$\mathrm{edge}(A)=\emptyset$

2) $B \subset \mathbb{R}^n$, $A := \{0\} \times B \subset \mathbb{R}^{n+1}$

then $\mathrm{edge}(A) = \{0\} \times \partial B$.