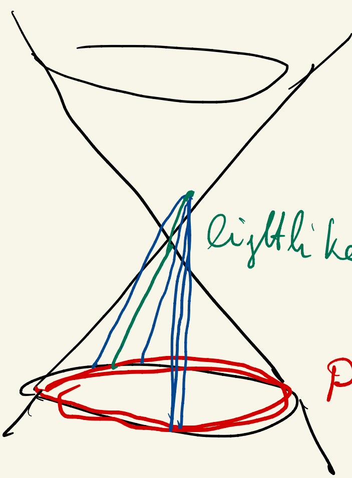
Prop 2.38 Let M be a Lorentzian mfd, $P \subset M$ a spacelike submanifold and $c : [0,b] \to M$ a lightlike geodesic in M with $c(0) = p \in P$ and $\dot{c}(0) \in N_p P$. Set $q := c(b)$.
If P has a focal point along c before q, that is for some $t \in (0,b)$, then in every nbhd of c (w.r.t. the compact-open topology), there is a timelike curve from P to q.

Example



Light cone in $\mathbb{R}^{n,1}$

$P = \{-1\} \times S^{n-1}$

$$\gamma(t) = \begin{pmatrix} (t-1) \\ (t-1)\vec{x} \end{pmatrix}$$

$\vec{x} \in S^{n-1}$

timelike

Proof: Let $t_0 > 0$ be the smallest focal value along c wrt. P. Let J be a Jacobi v.f. with $J(0) \in T_{c(0)}P$, $\pi^{tan}\left(\frac{DJ}{dt}(0)\right) = -S_{\dot{c}(0)}(J(0))$, $J(t_0)=0$, $J|_{(0,t_0)} \neq$

We already have $\boxed{\text{Claim 2}}$:

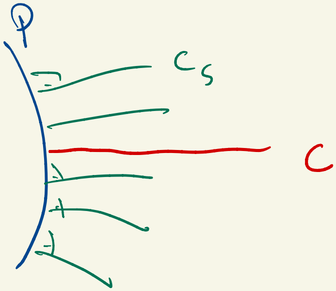
There is some $\delta \in (0, b-t_0)$ and a vector field V along c with $V(0)=J(0)$ and $V(t_0+\delta)=0$ such that $V \perp \dot{c}$ on $[0,t_0+\delta]$ and $\langle \frac{D^2}{dt^2}V + R(V,\dot{c})\dot{c}, V\rangle > 0$ on $(0,t_0+\delta)$.

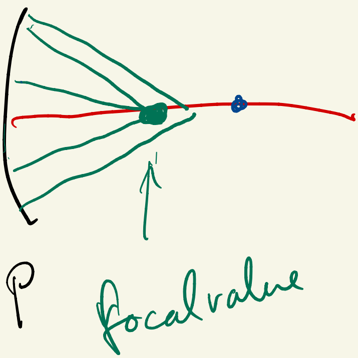
We will now prove $\boxed{\text{Claim 3}}$

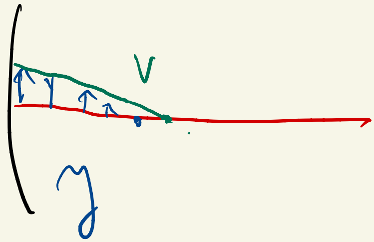
There is a smooth vector field X along $c|_{[0,t_0+\delta]}$ such that

$X(0) = \overrightarrow{\mathrm{II}}(J(0), J(0))$, $X(t_0+\delta) = 0$,

and on $[0,t_0+\delta]$: $\frac{d}{dt}\left(\langle V, \frac{DV}{dt}\rangle + \langle X, \dot{c}\rangle\right) \leq 0.$







Pf of Claim 3

$$\langle \mathrm{II}(\gamma(0), \gamma(0)), \dot c(0)\rangle = \langle S_{\dot c(0)}(\gamma(0)), \gamma(0)\rangle$$

$$= -\langle \cancel{a^{tan}}\left(\frac{D\gamma}{dt}(0)\right), \underbrace{\gamma(0)}_{\in T_pP}\rangle = -\langle \frac{D\gamma}{dt}(0), \gamma(0)\rangle$$

$$\langle V, \frac{DV}{dt}\rangle = ?$$

<u>Recall</u> $V = \gamma + hU \qquad \Big|\ \gamma = f \cdot U$

$= (f+h)U. \qquad h, f \in C^\infty([a,b])$

$U \in \Gamma(c^*TM), \ \langle U, U \rangle \equiv 1.$

$(f+h)(t_0+\delta) = 0$

$h = \beta(e^{\alpha t} - 1); \ \beta > 0$

$$\langle V, \frac{DV}{dt} \rangle = \langle \underbrace{\gamma + \beta(e^{\alpha t} - 1)U}_{V}, \frac{D\gamma}{dt} + \beta \alpha e^{\alpha t} U + \beta(e^{\alpha t} - 1)\frac{DU}{dt} \rangle$$

$$= \langle \gamma, \frac{D\gamma}{dt} \rangle + \langle V, \beta\alpha e^{\alpha t} U \rangle$$

$$+ \cancel{(\beta(e^{\alpha t} - 1))^2 \underbrace{\langle U, \frac{DU}{dt} \rangle}_{=0}} +$$

$$+\beta\left(e^{\alpha t}-1\right) \langle u, \frac{Dy}{dt}\rangle$$

$$+\beta\left(e^{\alpha t}-1\right)\underbrace{\cancel{\langle \frac{Du}{dt}, y(t)\rangle}}_{=0}$$

$$y(t) \parallel u(t)$$

$$y(0) = \|y(0)\| \cdot u$$

$$= \langle y, \frac{Dy}{dt}\rangle + \beta\alpha \overbrace{e^{\alpha t}}^{=1} \langle v, u\rangle$$

$$+\beta\underbrace{\left(e^{\alpha t}-1\right)}_{=0}\langle u, \frac{Dy}{dt}\rangle$$

$t=0$

$$\langle u(0), \frac{Du}{dt}\Big|_{t=0}\rangle = \langle y(0), \frac{Dy}{dt}(0)\rangle$$

$$+\beta\alpha\|y(0)\|$$

$$= -\underbrace{\langle \vec{\mathrm{II}}(\dot\gamma(0), \dot\gamma(0)), \dot c(0) \rangle}_{=: -a}$$

$$+ \alpha \beta \| \dot\gamma(0) \|$$

1st case $a \neq 0$

$\exists L_0 \in N_p P$ s.th.

$$\vec{\mathrm{II}}(\dot\gamma(0), \dot\gamma(0)) = a L_0$$

with $\langle L_0, \dot c(0) \rangle = -1$.

Assume $L \in \Gamma(c^* TM)$ with $\frac{DL}{dt} = 0$ and $L(0) = L_0$.

$$\frac{d}{dt} \langle L(t), \dot c(t) \rangle = 0 \Rightarrow \langle L(t), \dot c(t) \rangle \equiv -1$$

Set $$X(t) := \left(\left\langle V(t), \frac{\nabla V}{dt}(t) \right\rangle + \frac{\alpha\beta \|\gamma(0)\|}{t_0+\delta}(t - t_0 - \delta) \right) L(t)$$

<u>Thus</u>

$$X(0) = \left(a + \alpha\beta \cancel{\|\gamma(0)\|} + \cancel{\overbrace{\frac{-t_0-\delta}{t_0+\delta}}^{-1}} \cdot \alpha\beta \cancel{\|\gamma(0)\|} \right) L(0) = a L_0$$

$$= \vec{II}(\gamma(0), \gamma(0))$$

$$X(t_0+\delta) = \langle \underbrace{V(t_0+\delta)}_{=0}, \frac{DV}{dt}(t_0+\delta)\rangle$$

$$L(t_0+\delta) = 0.$$

$$\frac{d}{dt}\left(\langle V, \frac{DV}{dt}\rangle + \langle X, \dot{c}\rangle\right)$$

$$= \frac{d}{dt}\left(\langle V, \cancel{\frac{DV}{dt}}\rangle + \cancel{\langle V, \frac{DV}{dt}\rangle} \underbrace{\langle L, \dot{c}\rangle}_{=-1}\right)$$

$$+ \frac{\alpha \beta\, u\, y(0)}{t_0+\delta} \cdot (-1) \leq 0$$

2nd case

$a = 0$:

$$\langle \vec{\underline{\Pi}}(\dot\gamma(0), \dot\gamma(0)), \dot c(a) \rangle = 0$$

Choose $L, Z \in \Gamma(c^* TM)$

$\frac{\nabla}{dt} L = 0 = \frac{\nabla}{dt} Z$ with

$\langle L, \dot c \rangle = -1$, $Z(0) = \vec{\underline{\Pi}}(\dot\gamma(0), \dot\gamma(0))$

Thus $\langle Z(0), \dot c(0) \rangle = 0$

$\Rightarrow \langle Z, \dot c \rangle = 0$

$$X(t) := \Big(\langle V(t), \frac{\nabla V}{dt}(t) \rangle + \frac{\alpha\beta \|\dot\gamma(0)\|}{t_0 + \delta} \cdot$$

$$(t - t_0 - \delta)\Big) L(t) + \left(1 - \frac{t}{t_0 + \delta}\right) Z(t)$$

$$x(0) = \Big(0 + \cancel{\alpha\beta\|y(0)\|}$$

$$+ \cancel{\frac{-t_0-\delta}{t_0+\delta}\alpha\beta}\|y(0)\|\Big) L(0)$$

$$+ z(0) = \overrightarrow{\Pi}\big(y(0), J(0)\big)$$

$$x(t_0+\delta) = \Big(\langle \underbrace{V(t_0+\delta)}_{=0}, \frac{\nabla V}{dt}(t_0+\delta)\rangle$$

$$+ 0\Big) L(t_0+\delta) + 0 = 0.$$

$$\frac{d}{dt}\left(\langle V, \frac{\nabla V}{dt}\rangle + \langle X, \dot{c}\rangle\right) \qquad Z \perp \dot{c}$$

$$= \frac{d}{dt}\Big(\langle V, \cancel{\frac{\nabla V}{dt}}\rangle - \langle V, \cancel{\frac{\nabla V}{dt}}\rangle \qquad 0$$

$$V_{\parallel}$$

$$- \alpha\beta\|y(0)\|(t_0+\delta)^{-1} + \cancel{\langle \frac{Z(t)}{t_0+\delta}, \dot{c}(t)\rangle}$$

Proof of Prop 2.38

Choose a curve $\gamma : (-\varepsilon, \varepsilon) \to P$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = \gamma(0)$
$= V(0)$

$$\frac{\nabla^P \dot{\gamma}}{dt}(0) = 0 \qquad \text{(e.g. choose a geodesic in } P\text{)}$$

$$\frac{\nabla^M}{dt}\dot{\gamma}\Big|_{t=0} = \underbrace{\frac{\nabla^P}{dt}\dot{\gamma}}_{=0} + \pi^{nor}\left(\frac{\nabla^M}{dt}\dot{\gamma}\right)$$

$$= \vec{\mathrm{II}}(\dot{\gamma}(0), \dot{\gamma}(0)) = X(0)$$

<u>Exercise</u> There is a variation c_s of c with $(c_0 = c)$

$$c_s(0) = \gamma(s), \quad c_s(t_0+\delta) = c(t_0+\delta),$$

$$\frac{\partial \dot c_s}{\partial s}\Big|_{s=0} = V, \quad \frac{\nabla}{ds}\frac{\partial \dot c_s}{\partial s}\Big|_{s=0} = X$$

As c is lightlike, $\langle \dot c_s, \dot c_s \rangle\big|_{s=0} = 0$.

$$\frac{1}{2}\partial_s \langle \dot c_s, \dot c_s \rangle\Big|_{s=0} = \left\langle \frac{\nabla \dot c_s}{\partial s}, \dot c_s \right\rangle\Big|_{s=0}$$

$$= \Big\langle \frac{\nabla}{dt} \underbrace{\frac{\partial c_s}{\partial s}\Big|_{s=0}}_{V(t)}, \dot c \Big\rangle = \frac{d}{dt}\langle V(t), \dot c(t)\rangle$$

$$- \cancel{\Big\langle V(t), \overset{\overset{0}{\shortparallel}}{\frac{\nabla}{dt}\dot c(t)} \Big\rangle}$$

$V \perp \dot{c}$
$= 0$

$$\frac{1}{2} \frac{\partial^2}{\partial s^2} \langle \dot{c}_s, \dot{c}_s \rangle \Big|_{s=0} =$$

$$\frac{\partial}{\partial s}\Big|_{s=0} \langle \frac{D}{\partial s} \frac{\partial c_s}{\partial t}, \dot{c}_s \rangle$$

$$= \frac{\partial}{\partial s}\Big|_{s=0} \left(\langle \frac{D}{dt} V, \dot{c}_s \rangle \right)$$

$$= \langle \frac{D}{dt} V, \frac{D}{dt} V \rangle + \langle \frac{D}{ds} \frac{D}{dt} V, \dot{c}_s \rangle$$

$$= \langle \frac{D}{dt} V, \frac{D}{dt} V \rangle + \langle \frac{D}{dt} \frac{D}{ds} V, \dot{c}_s \rangle \Big|_{s=0}$$

$$+ \langle R(V, \dot{c}) V, \dot{c} \rangle$$

$$\boxed{\frac{\nabla}{ds}\frac{\partial c_s}{\partial s} = X.}$$

$$\frac{d}{dt}\langle X, \mathring{c}\rangle$$

$$= \left\langle \frac{\nabla X}{dt}, \mathring{c} \right\rangle + \frac{d}{dt}\left(\left\langle V, \frac{\nabla V}{dt}\right\rangle\right)$$

$$- \left\langle V, \frac{\nabla^2 V}{dt^2}\right\rangle - \langle R(V, \mathring{c})$$

$$\mathring{c}, V\rangle$$

Claim 3 ≤ 0

$$= \frac{d}{dt}\left(\langle X, \mathring{c}\rangle + \left\langle V, \frac{\nabla V}{dt}\right\rangle\right)$$

$$- \left\langle V, \frac{\nabla^2 V}{dt^2} + R(V, \mathring{c})\mathring{c}\right\rangle$$

Claim 2 ≤ 0

< 0

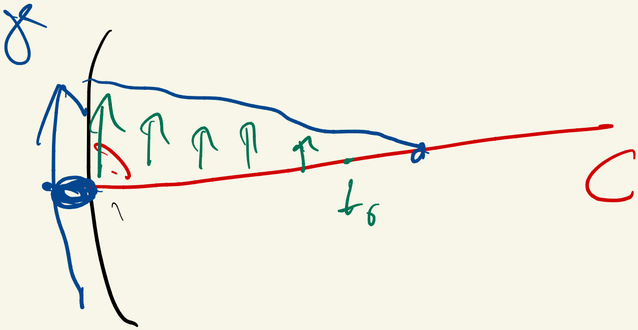
It follows $\langle \mathring{c}_s, \mathring{c}_s \rangle < 0$

for $s \in (0, s_0]$ for some

$s_0 > 0$

$\Longrightarrow \mathring{c}_s$ is timelike $\forall s \in (0, s_0]$

$\square$



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