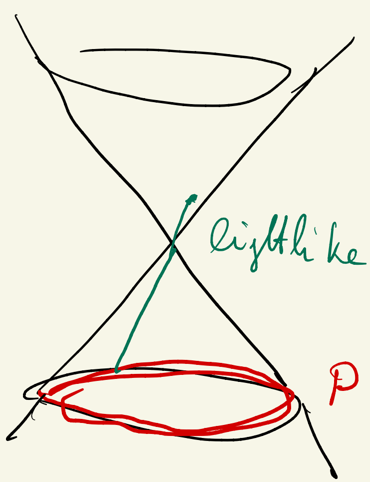
Prop 2.38 Let M be a Lorentzian mfd, $P \subset M$ a spacelike submanifold and $c : [0,b] \to M$ a lightlike geodesic in M with $c(0) = p \in P$ and $\dot{c}(0) \in N_p P$. Set $q := c(b)$.
If P has a focal point along c before q, that is for some $t \in (0,b)$, then in every nbhd of c (w.r.t. the compact-open topology), there is a timelike curve from P to q.

Example



Lightcone in $\mathbb{R}^{n,1}$

$P = \{-1\} \times S^{n-1}$

$$\gamma(t) = \begin{pmatrix} (t-1) \\ (t-1)\vec{x} \end{pmatrix}$$

$\vec{x} \in S^{n-1}$

Proof: Let $t_0 > 0$ be the smallest focal value along c wrt. P. Let Y be a Jacobi v.f. with $Y(0) \in T_{c(0)}P$, $\pi^{\tan}\left(\frac{DY}{dt}(0)\right) = -S_{\dot{c}(0)}(Y(0))$, $Y(t_0) = 0$

We already have $\boxed{\text{Claim 2}}$:

There is some $\delta \in (0, b - t_0)$ and a vector field V along c with $V(0) = Y(0)$ and $V(t_0 + \delta) = 0$ such that $V \perp \dot{c}$ on $[0, t_0 + \delta]$ and $\langle \frac{D^2}{dt^2} V + R(V, \dot{c})\dot{c}, V \rangle > 0$ on $(0, t_0 + \delta)$.

We will now prove $\boxed{\text{Claim 3}}$

There is a smooth vector field X along $c|_{[0, t_0 + \delta]}$ such that

$$X(0) = \overrightarrow{\mathrm{II}}(Y(0), Y(0)), \quad X(t_0 + \delta) = 0,$$

and on $[0, t_0 + \delta]$: $\frac{d}{dt}\left(\langle V, \frac{DV}{dt} \rangle + \langle X, \dot{c} \rangle\right) \leq 0.$