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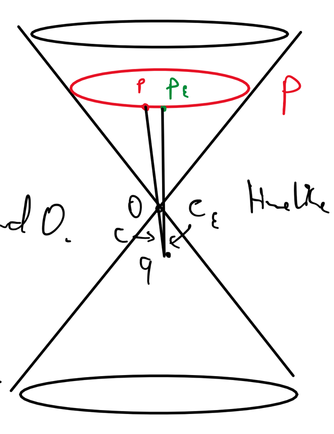
Prop 3.2.38 (M,g) Lor. mfd, $P \subseteq M$ spacelike subмfd., $c : [0,b] \to M$ lightlike geod.
$c(0) = p$, $\dot c(0) \in N_p P$, $q := c(b)$

If P has a focal pt along c before q (i.e. for $t \in (0,b)$), then every nbhd of c in cpt-open top. contains a timelike curve from P to q.

Example 3.2.39 $M = \mathbb{R}^{2,1}$, $P = \{1\} \times S^1 \ni p = \begin{pmatrix}1\\1\\0\end{pmatrix}$



$c(t) = (1-t)p$ lightlike geod through p and 0.

Let $q = \begin{pmatrix}\beta\\\beta\\0\end{pmatrix}$ for $\beta = 1-b$, $b > 1$ (so that $\beta < 0$).

$p_\varepsilon = \begin{pmatrix}1\\\cos(\varepsilon)\\\sin(\varepsilon)\end{pmatrix}$ for small $\varepsilon > 0$

$$\langle\langle q - p_\varepsilon, q - p_\varepsilon \rangle\rangle = \left\langle\left\langle \begin{pmatrix}\beta-1\\\beta-\cos(\varepsilon)\\-\sin(\varepsilon)\end{pmatrix}, \begin{pmatrix}\beta-1\\\beta-\cos(\varepsilon)\\-\sin(\varepsilon)\end{pmatrix} \right\rangle\right\rangle = -(\beta-1)^2 + (\beta-\cos(\varepsilon))^2 + \sin(\varepsilon)^2$$

$$= \cancel{-\beta^2} + 2\beta \cancel{-1} \cancel{+\beta^2} - 2\cos(\varepsilon)\beta \cancel{+1}$$

$$= \underbrace{2\beta}_{<0}\underbrace{(1-\cos(\varepsilon))}_{>0} < 0 \quad \rightsquigarrow \text{ straight line } c_\varepsilon \text{ is timelike.}$$

(Easy: $c_\varepsilon \longrightarrow c$ in compact-open top.)

Lemma 3.2.40 (M,g) semi-Riem. mfd, $P \subseteq M$ semi-Riem. subмfd., $c : [0,b] \to M$ geod, $c(0) = p$, $\dot c(0) \in N_p P$

Then $\mathcal{J} := \{t \in [0,b] \mid P$ has a focal pt along c at $t\} \subseteq (0,b]$ is compact.
In particular, there ex. a minimal value $t_0 := \min \mathcal{J} > 0$, the first focal value.

pf: $\mathcal{V} := \{\text{Jacobi fields } J \mid J(0) \in T_p P,\ \pi^{\tan}(\frac{\nabla J}{dt}(0)) = -S_{\dot c(0)} J(0)\}$ n-dim v.s., $n = \dim(M)$
(If t is a focal value, there ex. $J \in \mathcal{V} \setminus \{0\}$ s.th. $J(t) = 0$)

Equip this with $\|J\|_{C^1} = \sup_{t \in [0,b]} |J(t)|_h + \sup_{t \in [0,b]} |\frac{\nabla J}{dt}(t)|_h$, where h is some fixed Riemannian metric on M.
$\rightsquigarrow$ $(\mathcal{V}, \|\cdot\|_{C^1})$ normed finite dim v.s.

We show $\mathcal{J}$ is closed in $[0,b]$:
Let $(t_i)_{i \in \omega}$ be a seq. in $\mathcal{J}$, $t_i \longrightarrow t$. (to show: $t \in \mathcal{J}$).
Choose $J_i \in \mathcal{V} \setminus \{0\}$ s.th. $J_i(t_i) = 0$, wlog $\|J_i\|_{C^1} = 1$.
By compactness of $\{J \mid \|J\|_{C^1} = 1\} \subseteq \mathcal{V}$ (as $\mathcal{V}$ is finite-dim.), we may assume wlog that $J_i \xrightarrow{i \to \infty} J \in \mathcal{V}$. Note: $\|J\|_{C^1} = 1$, in part, $J \not\equiv 0$.

Let $P^{\nabla,c}_{s,t} : T_{c(s)}M \longrightarrow T_{c(t)}M$ be the parallel transport along c wrt LCC of (M,g).

$$|J(t)|_h = |J(t) - P^{\nabla,c}_{t_i,t} J(t_i) + P^{\nabla,c}_{t_i,t} J(t_i) - P^{\nabla,c}_{t_i,t} \underbrace{J_i(t_i)}_{=0}|_h$$

$$\le \underbrace{|J(t) - P^{\nabla,c}_{t_i,t} J(t_i)|_h}_{\longrightarrow 0 \text{ for } t_i \to t} + \underbrace{|P^{\nabla,c}_{t_i,t}(J(t_i) - J_i(t_i))|_h}_{\le C \underbrace{\|J - J_i\|_{C^1}}_{\to 0 \text{ for } i \to \infty} \text{ for some const. } C > 0 \text{ dep. on } h, c, P} \longrightarrow 0 \text{ for } i \to \infty$$

$$\Rightarrow \quad J(t) = 0 \quad \Rightarrow \quad t \in \mathcal{J} \cup \{0\}$$

Assume for contradiction that $t = 0$.

Then $J(0) = 0$, $\pi^{\tan}(\frac{\nabla J}{dt}(0)) = -S_{\dot c(0)} J(0) = 0$

Moreover,
$$\left|\pi^{\mathrm{nor}}\left(\frac{P^{\nabla,c}_{t_i,0} J(t_i)}{t_i}\right)\right|_h = \left|\pi^{\mathrm{nor}}\left(\underbrace{\frac{P^{\nabla,c}_{t_i,0} J(t_i) - J(0)}{t_i - 0}}_{\to \frac{\nabla J}{dt}(0)}\right)\right|_h \longrightarrow \left|\pi^{\mathrm{nor}}\left(\frac{\nabla J}{dt}(0)\right)\right|_h$$

$$\left\|$$

$$\left|\pi^{\mathrm{nor}}\left(\frac{P^{\nabla,c}_{t_i,0} J(t_i) - P^{\nabla,c}_{t_i,0} \overbrace{J_i(t_i)}^{=0}}{t_i}\right)\right|_h = \frac{1}{t_i}\left|\pi^{\mathrm{nor}}\left(\underbrace{J(0)}_{=0} + \int_0^{t_i} P^{\nabla,c}_{\tau,0} \frac{\nabla J}{dt}(\tau)\,d\tau - \underbrace{J_i(0)}_{\in T_p P} - \int_0^{t_i} P^{\nabla,c}_{\tau,0} \frac{\nabla J_i}{dt}(\tau)\,d\tau\right)\right|_h$$

$$= \frac{1}{t_i}\left|\pi^{\mathrm{nor}}\left(\int_0^{t_i} P^{\nabla,c}_{\tau,0}\left(\frac{\nabla(J - J_i)}{dt}(\tau)\right) d\tau\right)\right|_h$$

$$\le \frac{1}{t_i}\int_0^{t_i} \left|\pi^{\mathrm{nor}}\left(P^{\nabla,c}_{\tau,0} \frac{\nabla(J-J_i)}{dt}(\tau)\right)\right|_h d\tau \le \frac{\hat C}{t_i}\int_0^{t_i} \|J - J_i\|_{C^1}\,d\tau$$

$$= \hat C \|J - J_i\|_{C^1} \xrightarrow{i \to \infty} 0$$

$$\Rightarrow \quad \pi^{\mathrm{nor}}\left(\frac{\nabla J}{dt}(0)\right) = 0 \quad \Rightarrow \quad J(0) = 0,\ \frac{\nabla J}{dt}(0) = 0 \quad \Rightarrow \quad J \equiv 0 \ ↯$$ □

proof of 3.2.38: Let t_0 be the first focal value in $[0,b]$, $J \in \mathcal{V} \setminus \{0\}$ be a ass. Jacobi field with $J(t_0) = 0$. Note: $J(t) \neq 0$ $\forall t \in (0, t_0)$.

Claim 1: $\exists \delta \in (0, b - t_0)$: $J = f \cdot U$ on $[0, t_0 + \delta]$, where U is a spacelike unit normal field along c, $f \in C^\infty([0, t_0+\delta], \mathbb{R})$, $f > 0$ on $(0, t_0)$, $f < 0$ on $(t_0, t_0 + \delta)$

pf: First, show that J is spacelike on $(0, t_0)$.
We have $J \perp \dot c$ on $[0,b]$ (on new exercise sheet)
⚠ We may still have $J \parallel \dot c$ somewhere, as $\dot c$ is lightlike!
Assume for contr. that $\exists t_1 \in (0, t_0)$: $J(t_1) = a \dot c(t_1)$.
$J_0(t) := \frac{at}{t_1} \dot c(t)$ is a Jacobi field with $J_0(0) = 0$, $\frac{\nabla J_0}{dt}(0) = \frac{a}{t_1} \dot c(0) \in N_p P$
$\rightsquigarrow J_0 \in \mathcal{V}$
$\Rightarrow \tilde J = J - J_0 \in \mathcal{V}$ with $\tilde J(t_1) = 0$
$\overset{t_0 \text{ first focal value}}{\Rightarrow}$ $\tilde J \equiv 0$ $\Rightarrow$ $0 = J(t_0) = J_0(t_0) = a \frac{t_0}{t_1} \dot c(t) \neq 0$ ↯

So: $J \perp \dot c$, $J \not\parallel \dot c$, $J \neq 0$ for all $t \in (0, t_0)$ $\Rightarrow$ $J(t)$ spacelike on $(0, t_0)$.

Let $\varphi(t) = \begin{cases} t(t_0 - t) & \text{if } J(0) = 0 \\ t_0 - t & \text{if } J(0) \neq 0 \end{cases}$. Then $Y = \frac{J}{\varphi}$ extends to a smooth vf along $c|_{[0,t_0]}$.

Y is spacelike on $(0, t_0)$. This is also true on $[0, t_0]$:
Look at $Y(0)$: Case $J(0) \neq 0$: $T_p P \ni \underset{\frac{1}{t_0} J(0)}{\overset{\shortparallel}{Y(0)}} \neq 0$ spacelike

Case $J(0) = 0$: $Y(0) = \frac{1}{t_0} \frac{\nabla J}{dt}(0) \neq 0$ (otherwise $J \equiv 0$)

$0 = \frac{d}{dt} \underbrace{\langle J, \dot c \rangle}_{\equiv 0} = \langle \frac{\nabla J}{dt}, \dot c \rangle \Rightarrow Y(0) \perp \dot c(0)$

If $Y(0) = a \dot c(0)$, then $\frac{\nabla J}{dt}(0) = \varphi'(0) Y_0 = a t_0 \dot c(0)$
$\Rightarrow J(t) = a t_0 \dot c(t)$ ↯ $J(t_0) = 0$
$\Rightarrow Y(0) \not\parallel \dot c(0)$

$\Rightarrow Y(0)$ spacelike

That $Y(t_0)$ is spacelike uses an argument analogous to $(J(0) = 0)$-case.

$\Rightarrow$ Y nowhere vanishing, smooth spacelike on $[0, t_0]$ $\overset{\text{continuity}}{\Rightarrow}$ still true on $[0, t_0 + \delta]$ for small $\delta > 0$.

Now set $U = \frac{Y}{\|Y\|}$, $f = \varphi \cdot \|Y\|$. □ Claim 1

Claim 2: $\exists \delta \in (0, b - t_0)$, V vf. along c: $V(0) = J(0)$, $V(t_0 + \delta) = 0$
$V \perp \dot c$ on $[0, t_0 + \delta]$
$\langle \frac{\nabla^2 V}{dt} + R(V, \dot c)\dot c, V \rangle > 0$ on $(0, t_0 + \delta)$

pf: Choose δ, f, U as in Claim 1.
Ansatz: $V := (f + h) U = J + hU$ for $h \in C^\infty([0, t_0+\delta], \mathbb{R})$.

$$\frac{\nabla^2 V}{dt^2} + R(V, \dot c)\dot c \overset{J \text{ Jacobi field}}{=} h'' U + 2h' \frac{\nabla U}{dt} + h\left(\frac{\nabla^2 U}{dt^2} + R(U, \dot c)\dot c\right)$$

$$\left\langle \frac{\nabla^2 V}{dt^2} + R(V, \dot c)\dot c, V \right\rangle = (f + h)\Big(h'' \underbrace{\langle U, U \rangle}_{\equiv 1} + 2h' \underbrace{\langle \frac{\nabla U}{dt}, U \rangle}_{\equiv 0} + h \underbrace{\langle \frac{\nabla^2 U}{dt^2} + R(U, \dot c)\dot c, U \rangle}_{=: \ell(t)}\Big)$$

$$= (f + h)(h'' + h\ell)$$

ℓ bdd on $[0, t_0 + \delta]$ $\Rightarrow$ $\exists \alpha > 0$: $\ell \geq -\alpha^2$.

Set $h(t) := \beta(e^{\alpha t} - 1)$ with $\beta = \frac{-f(t_0 + \delta)}{e^{\alpha(t_0+\delta)} - 1} > 0$.

Then: $h(t_0 + \delta) = -f(t_0 + \delta)$, so that $V(t_0 + \delta) = 0$

$h(0) = 0 \Rightarrow V(0) = J(0)$

$h + f > 0$ on $(0, t_0]$ and we may assume wlog (possibly making δ smaller) that δ is the first zero after t_0 of $f + h$.
$\Rightarrow h + f > 0$ on $(0, t_0 + \delta)$

$\Rightarrow$ On $(0, t_0 + \delta)$: $\underbrace{(f + h)}_{>0}(h'' + h\ell) > 0$ since

$h'' + h\ell = \alpha^2 h(t) + \alpha^2 \beta + \ell(t) h(t) \geq \alpha^2 \beta > 0$ □ Claim 2