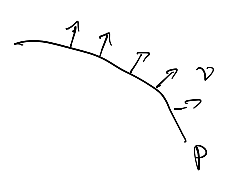
Montag, 7. Juni 2021 08:15

Recall from last lecture:

- Jacobi field equation $\frac{\nabla^2 J}{dt^2} + R(J, \dot c)\dot c = 0$ for a vf J along a geodesic c has $2\cdot\dim(M)$ dimensional space of solutions $\leftrightsquigarrow$ geod. variations of c with var. vf. J
- On model spaces, Jacobi fields are given by generalized sine and cosine fctns.

Today: Focal points.

Let $P \subseteq M$ be a semi-Riem. submfd, $\nu \perp TP$ normal vector ($\nu \in NP = (TP)^\perp \subseteq TM|_P$)



Shape operator w.r.t. ν: $S_\nu(X) := g(\nu, \vec{II}(X, -))^\# \in TP$

i.e. $g(S_\nu(X), Y) = g(\nu, \vec{II}(X,Y)) \quad \forall Y \in T_pP, X \in T_pP$

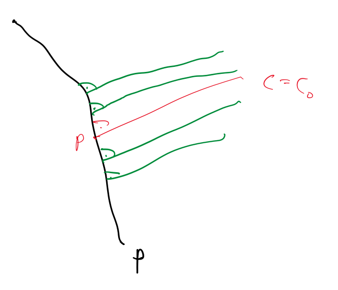
$\leadsto S_\nu : T_pP \longrightarrow T_pP.$

Lem 3.2.34 $P \subseteq M$ semi-Riem. submfd, $p \in P$, c geod. of M, $c(0) = p$, $\dot c(0) \in N_pP$ $(= (T_pP)^\perp)$

For a Jacobi field J, TFAE

(i) J is a var. vf of a geod. var. c_s of c with

$c_s(0) \in P \quad \forall s$
$\dot c_s(0) \in N_{c(s)}P \quad \forall s$



(ii) $J(0) \in T_pP$
$\pi^{tan}\left(\frac{\nabla J}{dt}(0)\right) = -S_{\dot c(0)}(J(0))$
(π^{tan}: orth. proj. $TM|_P \longrightarrow TP$)

Pf: (i) $\Rightarrow$ (ii) $\gamma(s) := c_s(0)$ is a curve in P

$\Rightarrow J(0) = \frac{d}{ds}\big|_{s=0} c_s(0) = \gamma'(0) \in T_pP$

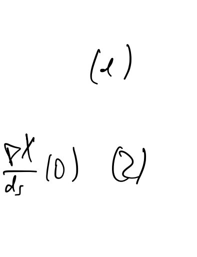
Let $X \in \mathfrak{X}(P)$ and show $g\left(\frac{\nabla J}{dt}(0), X\right) \overset{!}{=} g(-S_{\dot c(0)}J(0), X) = -g(\dot c(0), \vec{II}(J(0), X)) = -g(\dot c(0), \nabla_{J(0)}X)$

$0 = \underbrace{\frac{d}{ds}\big|_{s=0} g(X, \dot c_s(0))}_{\equiv 0} = g\left(\frac{\nabla(X\circ\gamma)}{ds}\big|_{s=0}, \dot c_0(0)\right) + g\left(X, \underbrace{\frac{\nabla}{ds}\big|_{s=0}\frac{dc_s}{dt}\big|_{t=0}}_{= \frac{\nabla}{dt}\big|_{t=0}\frac{dc_s(t)}{ds}\big|_{s=0} = \frac{\nabla J}{dt}(0)}\right)$

$= g(\nabla_{\gamma'(0)}X, \dot c(0)) + g\left(X, \frac{\nabla J}{dt}(0)\right) \quad \checkmark$ (with $\gamma'(0) = J(0)$)

(ii) $\Rightarrow$ (i) Choose a curve γ in P with $\gamma(0) = p$, $\gamma'(0) = J(0) \in T_pP$

We will define $c_s(t) := \exp_{\gamma(s)}(t \cdot X(s))$ for a suitable X along γ.



Denote by $\tilde J$ the var. vf. of c_s.

Note $\tilde J(0) = \frac{dc_s(0)}{ds}\big|_{s=0} = \gamma'(0) = J(0)$.

We must have

$\dot c(0) = \dot c_0(0) = \frac{d}{dt}\big|_{t=0} \exp_p(tX(0)) = X(0) \qquad (1)$

$\frac{\nabla J}{dt}(0) \overset{!}{=} \frac{\nabla \tilde J}{dt}(0) = \frac{\nabla}{dt}\big|_{t=0}\frac{dc_s(t)}{ds}\big|_{s=0} = \frac{\nabla}{ds}\big|_{s=0}\dot c_s(0) = \frac{\nabla}{ds}\big|_{s=0}X(s) = \frac{\nabla X}{ds}(0) \qquad (2)$

Consider the normal bundle $NP \longrightarrow P$ with connection $\nabla^{nor}_X Y := \pi^{nor}(\nabla_X Y)$ ($X \in T_pP$, $Y \in \Gamma(P, NP)$; Y vf. of TM along P)

Let U be the parallel transport of $\dot c(0)$ in (NP, ∇^{nor}) along γ
" V — " — $\pi^{nor}\left(\frac{\nabla J}{dt}(0)\right)$ — " —

Set $X(s) := U(s) + s \cdot V(s)$.

Then, we have $X(0) = U(0) = \dot c(0) \qquad (1)$

$\pi^{nor}\left(\frac{\nabla X}{ds}(0)\right) = \underbrace{\pi^{nor}\left(\frac{\nabla U}{ds}(0)\right)}_{= 0 \text{ by def of par. transport}} + \pi^{nor}(V(0)) = \pi^{nor}\left(\frac{\nabla J}{dt}(0)\right) \qquad \pi^{nor}\circ(2)$

$\pi^{tan}\left(\frac{\nabla X}{ds}(0)\right) = \pi^{tan}\underbrace{\left(\frac{\nabla U}{ds}(0)\right)}_{\in TP} + \underbrace{\pi^{tan}(V(0))}_{= 0} = \frac{\nabla U}{ds}(0) \qquad (3)$

Remains to show $\pi^{tan}\circ(2) \overset{(3)}{\Leftrightarrow} \frac{\nabla U}{ds}(0) = \underbrace{\pi^{tan}\left(\frac{\nabla J}{dt}(0)\right)}_{\overset{ass.}{=} -S_{\dot c(0)}(J(0))}$.

Let $Y \in \mathfrak{X}(P)$.

$g\left(\frac{\nabla U}{ds}(0), Y\right) \overset{U \perp Y}{=} -g(\underbrace{U(0)}_{normal}, \nabla_{\gamma'(0)}Y) = -g(\underbrace{U(0)}_{= \dot c(0)}, \vec{II}(\underbrace{\gamma'(0)}_{= J(0)}, Y))$

$= g(-S_{\dot c(0)}J(0), Y) \quad \checkmark$ □

Def 3.2.35

$P \subseteq M$ semi-Riem. submfd, $p \in P$, c geod. in M, $c(0) = p$, $\dot c(0) \in N_pP$

Then P has a focal point of order μ along c at t ($t \neq 0$) if

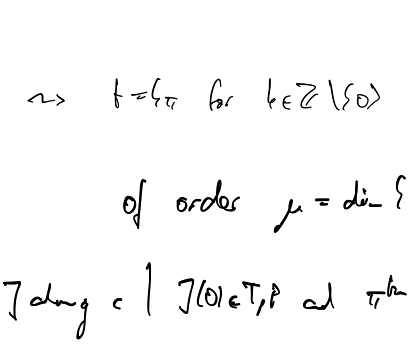
$\mu := \dim \left\{ \text{Jacobi fields } J \text{ along } c \;\middle|\; \begin{array}{l} J(0) \in T_pP \\ \pi^{tan}\left(\frac{\nabla J}{dt}(0)\right) = -S_{\dot c(0)}J(0) \\ J(t) = 0 \end{array} \right\} > 0$

In this case, t is called focal value of P along c.

If $P = \{p\}$ is a point, then a focal point is called conjugated point.

Examples 3.2.36

1. $M = S^n$, $P = \{p\}$
c geod. through p



We know that
$J(t) = \sin(t) E(t)$ for $E \perp \dot c$ is parallel along c
is a Jacobi field

$J(0) = 0 = J(k\pi)$ for $k \in \mathbb{Z}$ $\leadsto$ $t = k\pi$ for $k \in \mathbb{Z} \setminus \{0\}$ is a conjugated point of order $\mu = \dim\{E \perp \dot c\} = n-1$

Rmk 3.2.37 $\mathcal{V} := \{\text{Jacobi fields } J \text{ along } c \mid J(0) \in T_pP \text{ and } \pi^{tan}\left(\frac{\nabla J}{dt}(0)\right) = -S_{\dot c(0)}J(0)\}$ $\ni \tilde J(t) := t\,\dot c(t)$

$\dim \mathcal{V} = \dim \{(J, J') \in T_pM \times T_pM \mid J \in T_pP \text{ and } \pi^{tan}(J') = -S_{\dot c(0)}J\}$

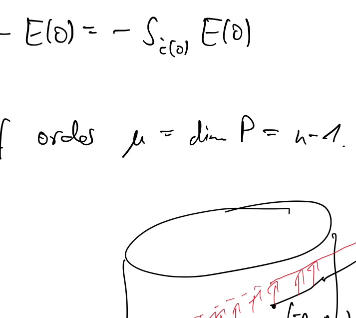
$= \dim(P) + (\dim(M) - \dim(P)) = \dim(M) =: n$

$\mu = \dim\{J \in \mathcal{V} \mid J(t) = 0\} \leq n-1$ ($\not\ni \tilde J$)

Upshot: The order is maximal in ex. 1.

2. $M = \mathbb{R}^n$, $P = S^{n-1} \ni p$

c geod. through p and 0
given by $c(t) = (1-t)\cdot p$



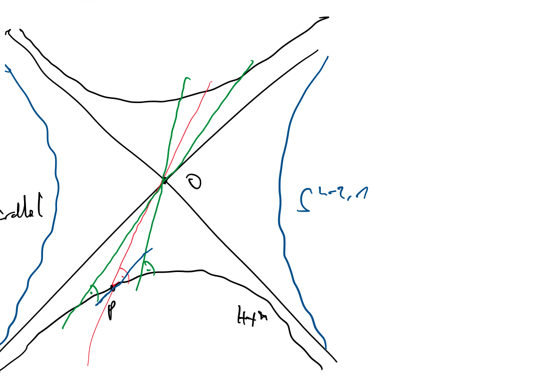
$J(t) = (1-t)E(t)$, where E parallel along c, $E(0) \in T_pP$ is a Jacobi field with

$\pi^{tan}\left(\frac{\nabla J}{dt}(0)\right) = \pi^{tan}(-E(0)) = -E(0) = -S_{\dot c(0)}E(0)$

As $J(1) = 0$, $t = 1$ is a focal pt of order $\mu = \dim P = n-1$.

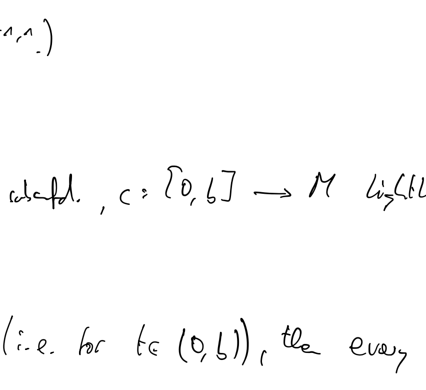
2'. $M = \mathbb{R}^n$, $P = S^{k-1} \times \mathbb{R}^{n-k} \ni (p_1, p_2)$

$c(t) := ((1-t)p_1, p_2)$



Similarly as in ex 2, get that
$t = 1$ is a focal pt of order $\mu = \dim S^{k-1} = k-1$ ($\leq n-1$)

3. $M = \mathbb{R}^{n-1,1}$, $P = \mathbb{H}^{n-1} \ni p$
$c(t) = (1-t)p$ geod. through p and 0



As in ex. 2, $J(t) = (1-t)E(t)$ with E parallel along c, $E(0) \in T_pP$
shows that $t = 1$ is a focal pt of order $\mu = n-1$.
(The same also works for $S^{n-2,1} \subseteq \mathbb{R}^{n-1,1}$.)

Prop 3.2.38 (M, g) Lor. mfd, $P \subseteq M$ spacelike submfd., $c : [0, b] \longrightarrow M$ lightlike geod.
$c(0) = p$, $\dot c(0) \in N_pP$, $q := c(b)$

If P has a focal pt along c before q (i.e. for $t \in (0, b)$), then every nbhd of c in cpt-open top. contains a timelike curve from p to q.