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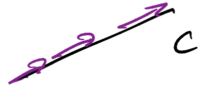


Example 2.33.

1. Let c be a geodesic and $J(t) := (at+b)\dot{c}(t)$ for some $a, b \in \mathbb{R}$. Since

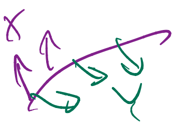
$$\frac{\nabla^2 J}{dt^2}(t) = \frac{\nabla}{dt}\left(a\dot{c}(t) + (at+b)\underbrace{\frac{\nabla \dot{c}}{dt}(t)}_{=0}\right) = a\frac{\nabla \dot{c}}{dt}(t) = 0 = (at+b)R(\dot{c},\dot{c})\dot{c} = R(\dot{c}, J)\dot{c},$$

   J is a Jacobi field with corresponding geodesic variation $c_s(t) = c\big((1+as)t+bs\big)$, i.e. a mere reparametrization of c.

2. For $M = \mathbb{R}^n$ the Euclidean or Minkowski space, the Jacobi equation reads $\frac{\nabla^2 J}{dt^2} = 0$ with solution $J(t) = tX(t) + Y(t)$ for any parallel vector fields X, Y. The corresponding geodesic variation of the straight line c is then

$$c_s(t) = c(t) + s\big(tX(t) + Y(t)\big).$$

3. For M having constant sectional curvature κ, we obtain



$$R(X,Y)Z = \kappa\big(g(Y,Z)X - g(X,Z)Y\big)$$

$$\frac{\nabla}{dt} X = 0$$

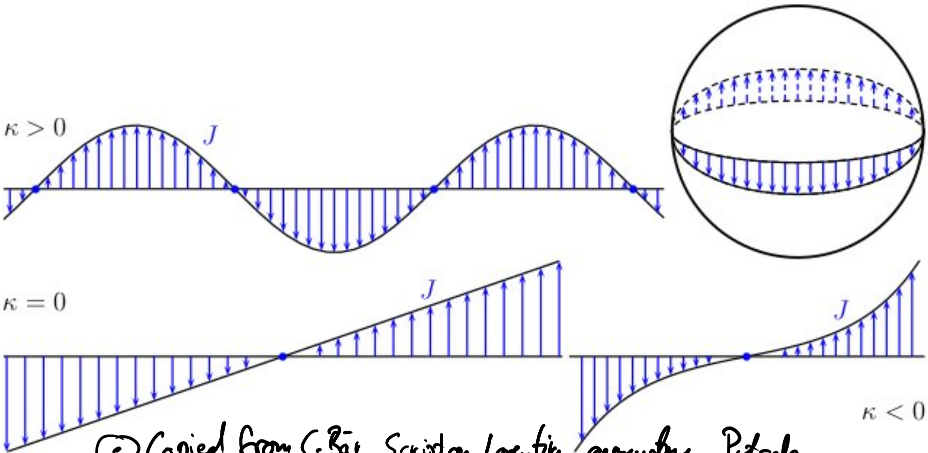
for vector fields X, Y, Z. Let c be a geodesic and X, Y parallel vector fields along c, which are pointwise orthogonal to $\dot{c}$. For $\delta \in \mathbb{R}$, we introduce the generalized sine and cosine function via

$$\mathfrak{s}_\delta(t) := \begin{cases} \frac{1}{\sqrt{\delta}} \sin\left(\sqrt{\delta} t\right), & \delta > 0, \\ t, & \delta = 0 \\ \frac{1}{\sqrt{|\delta|}} \sinh\left(\sqrt{|\delta|} t\right), & \delta < 0 \end{cases}, \quad \mathfrak{c}_\delta(t) := \begin{cases} \frac{1}{\sqrt{\delta}} \cos\left(\sqrt{\delta} t\right), & \delta > 0, \\ 1, & \delta = 0 \\ \frac{1}{\sqrt{|\delta|}} \cosh\left(\sqrt{|\delta|} t\right), & \delta < 0 \end{cases},$$

which then satisfy $\mathfrak{s}_\delta'' = -\delta \mathfrak{s}_\delta$ and $\mathfrak{c}_\delta'' = -\delta \mathfrak{c}_\delta$. Hence, $J(t) := \mathfrak{s}_{\eta\kappa}(t) X(t) + \mathfrak{c}_{\eta\kappa} Y(t)$, where $\eta := g(\dot{c}, \dot{c})$, fulfills $\frac{\nabla^2 J}{\mathrm{d}t^2} = -\eta\kappa J$. On the other hand, we have

$$R(\dot{c}, J)\dot{c} = \kappa\left(\underbrace{g(J, \dot{c})}_{=0} \dot{c} - \underbrace{g(\dot{c}, \dot{c})}_{=\eta} J\right) = -\eta\kappa J,$$

so J is a Jacobi field.



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