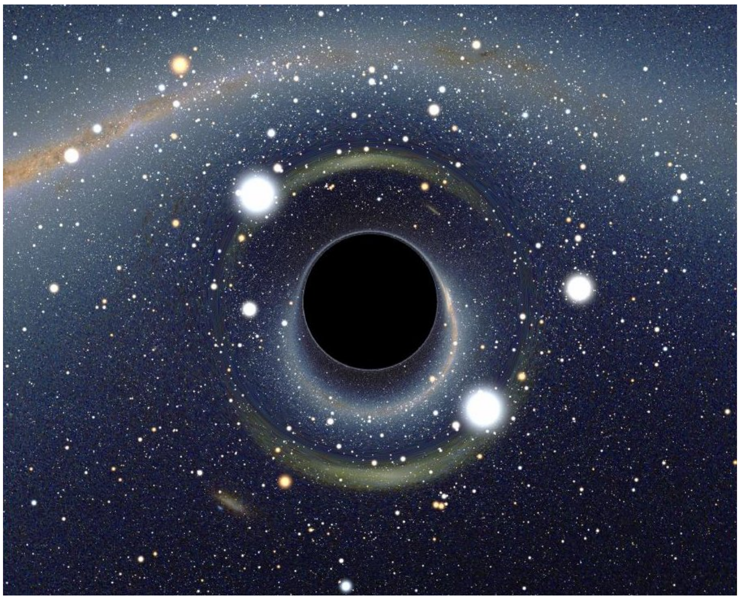
Differential Geometry II
Lorentzian Geometry
Lecture Notes



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Presentation Version

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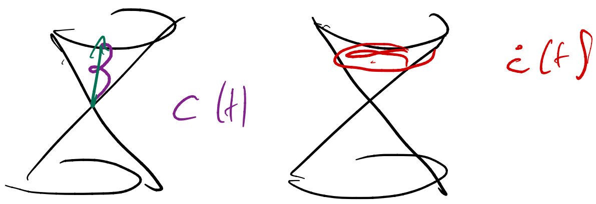
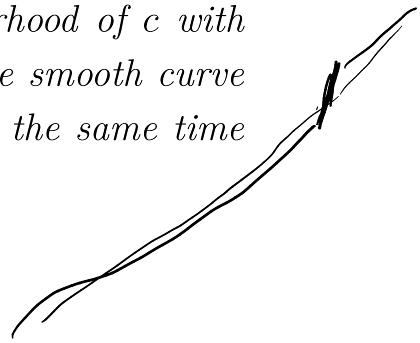
Lemma 3.2.29. *Let $c : [a,b] \to M$ be a causal piecewise C^1-curve which is not a lightlike pregeodesic. In each neighborhood of c with respect to the compact-open topology we find a timelike smooth curve with the same start and end point. The new curve has the same time direction as c.*





Proof:

(1) we can assume w. l. o. g. that c is smooth (see handwritten notes, follows from Cor. 3.2.17a)

(2) The statement follows if there is a $t_0 \in [a,b]$ with $\dot c(t_0)$ timelike. (see handwritten notes)

(3) We assume that c is a smooth lightlike curve, but not a pregeodesic. W. l. o. g. $[a,b] = [0,1]$. Deriving $g\big(\dot c(t), \dot c(t)\big) = 0$ we obtain

$$0 = \frac{d}{dt} g\big(\dot c(t), \dot c(t)\big) = 2g\Big(\frac{\nabla}{dt}\dot c(t), \dot c(t)\Big). \tag{3.2.1}$$

Thus

$$\frac{\nabla}{dt}\dot c(t) \in (\dot c(t))^{\perp} . = \mathbb{R}\,\dot c(t) \oplus \overset{\text{spacelike}}{E_t}$$

We have explained (see in the handwritten notes for details): as c is not a pregeodesic, there is some $t_0 \in [0,1]$ with

$$\frac{\nabla}{dt}\Big|_{t=t_0} \dot c(t) \notin \mathbb{R}\dot c(t_0)$$

and that this implies

$$g\left(\frac{\nabla}{dt}\Big|_{t=t_0} \dot c(t), \frac{\nabla}{dt}\Big|_{t=t_0} \dot c(t)\right) > 0$$

Deriving (3.2.1) once again we get

$$0 = \frac{d}{dt} g\Big(\frac{\nabla}{dt}\dot c(t), \dot c(t)\Big) = g\Big(\frac{\nabla}{dt}\frac{\nabla}{dt}\dot c(t), \dot c(t)\Big) + g\Big(\frac{\nabla}{dt}\dot c(t), \frac{\nabla}{dt}\dot c(t)\Big).$$

Choose $Y_0 \in T_{c(0)}M$ timelike

with $g(Y_0, \dot{c}(0)) < 0$.

Find $Y \in \Gamma(c^* TM)$ s.th. $Y(0) = Y_0$

and $\frac{D}{dt} Y = 0$.

Then $g(\underbrace{Y(t)}_{\text{timelike}}, \underbrace{\dot{c}(t)}_{\text{lightlike}}) \neq 0$ cont. in t

$\Rightarrow g(Y(t), \dot{c}(t)) < 0$.

<u>Ansatz</u>: Suppose we have

smooth functions $\alpha, \beta : [0,1] \to \mathbb{R}$

$\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0$ and

s.th. $X(t) := \alpha(t) Y(t) + \beta(t) \frac{D}{dt} \dot{c}(t)$

satisfies $\forall t \in [0,1]$ $g(\frac{\nabla X}{dt}, \dot{c}(t)) < 0$.

Suppose we have such α, β. $\rightsquigarrow X$ ■

Determine $c_\circ : [0,1] \times (-\varepsilon, \varepsilon) \to M$

$(t,s) \longmapsto c_s(t) = c_\circ(t,s)$

such $c_0 = c$ and $\left.\frac{\partial}{\partial s}\right|_{s=0} c_s(t) = X(t)$

$$\left.\frac{d}{ds}\right|_{s=0} \underbrace{g(\dot{c}_s(t), \dot{c}_s(t))}_{\text{for } S=0:\ 0} =$$

$$= 2\, g\left(\left.\frac{\nabla}{ds}\right|_{s=0} \frac{\partial}{\partial t} c_s(t), \dot{c}_0(t)\right)$$

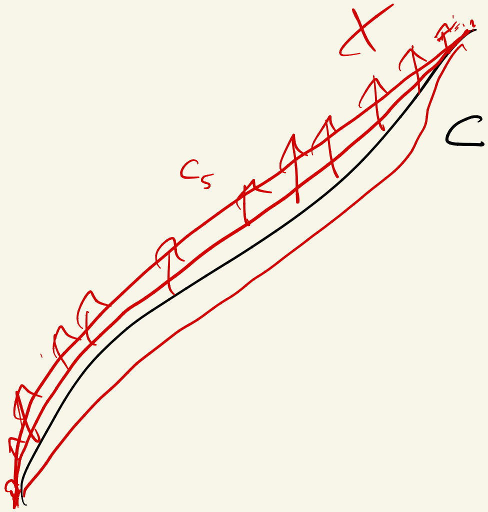
$$= 2\, g\Big(\frac{\nabla}{dt} \underbrace{\left.\frac{\partial}{\partial s}\right|_{s=0} c_s}_{X(t)}(t), \dot{c}(t)\Big) < 0$$

$$\Rightarrow \exists s_0 > 0 \ \forall s \in (0, s_0) \ \forall t \in [0,1]$$

$$g(\dot{c}_s(t), \dot{c}_s(t)) < 0$$

$\Rightarrow$ c_s is timelike $\forall s \in (0, s_0)$

$\Rightarrow$ statement


X
C_5
C

Proof of the Ansatz

$$g\left(\frac{D X}{dt}, \dot{c}\right) = \dot{\alpha} \quad g(Y, \dot{c})$$

$$+ \alpha \, g\Big(\underbrace{\frac{D}{dt} Y}_{=0}, \dot{c}\Big) + \dot{\beta} \, \overbrace{g\left(\frac{D}{dt} \dot{c}, \dot{c}\right)}^{=0}$$

$$+ \beta \, \underbrace{g\left(\frac{D}{dt}\frac{D}{dt} \dot{c}, \dot{c}\right)}_{< 0 \text{ at } t_0}$$

Then $\gamma(t) := \dfrac{g\left(\frac{D}{dt} \dot{c}(t), \frac{D}{dt} \dot{c}(t)\right)}{g(Y(t), \dot{c}(t))}$ $\underset{\substack{< 0 \\ \text{at } t = t_0}}{\leq 0}$

$$\gamma \not\equiv 0 .$$

Thus we determine $\beta : [0, 1] \to \mathbb{R}$ with $\int_0^1 \beta(t) \gamma(t) = -1$.

and $\beta(0) = \beta(1) = 0$.

Define $\alpha(t) := \int_0^t (\beta(t)\gamma(t) + 1)\, ds$

$\Rightarrow \alpha(0) = 0 = \alpha(1)$

$$g\left(\frac{DX}{dt}, \dot c\right) = \overbrace{(\beta\gamma + 1)}^{> \beta\gamma}\, \overbrace{g(Y, \dot c)}^{< 0}$$

$$+ \beta\, g\left(\frac{D}{dt}\frac{D}{dt}\dot c, \dot c\right)$$

$$< \beta\, g\left(\frac{D}{dt}\dot c, \frac{D}{dt}\dot c\right) + \beta\, g\left(\frac{D}{dt}\frac{D}{dt}\dot c, \dot c\right)$$

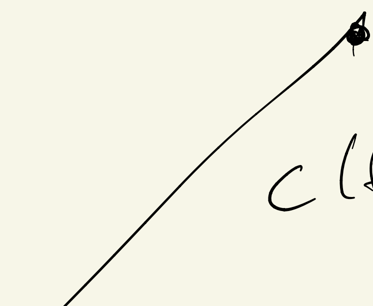
$$= 0.$$

Remark

If c is a lightlike pregeodesic, then, the conclusion of Lemma 2.29 does not hold.

$M = \mathbb{R}^{n,1}$ $\quad c(t) = t \cdot X \quad a<b$

X lightlike



$c(t)$

Only causal curve from $c(a)$ to $c(b)$ is up to reparametrization c.

Lemma 2.30

Assume that $c : [a,b] \to M$ is a lightlike smooth geodesic and $c_\cdot : [a,b] \times (-\varepsilon, \varepsilon) \to M$ a smooth variation $(t,s) \longmapsto c_s(t)$. Define variation v.f.

$$X = \frac{d}{ds}\Big|_{s=0} c_s \in \Gamma(c^* TM),$$

$\boxed{c_0 = c}$ and we assume

1.) $g(X(a), \dot{c}(a)) = 0$ and $g(X(b), \dot{c}(b)) = 0$

2.) $\exists\, s_i \to 0$ s.th. c_{s_i} is timelike $\forall i$

Then $g(\frac{\nabla}{dt} X(t), \dot{c}(t)) = 0 \quad \forall t \in [0,1]$.

Proof: Wlog $s_i > 0 \ \forall i$

$$0 \geq \lim_{i \to \infty} \frac{g(\dot{c}_{s_i}, \dot{c}_{s_i})}{s_i} = \lim_{i \to \infty} \frac{g(\dot{c}_{s_i}, \dot{c}_{s_i}) - g(\dot{c}_0, \dot{c}_0)}{s_i - 0}$$

$$= \frac{\partial}{\partial s}\Big|_{s=0} g(\dot{c}_s, \dot{c}_s)$$

$$\frac{\partial}{\partial s}\Big|_{s=0} g(\dot{c}_s, \dot{c}_s) = 2 g\left(\frac{D}{ds}\Big|_{s=0} \frac{\partial}{\partial t} c_s, \dot{c}_0\right)$$

$$= 2 g\Big(\frac{D}{dt} \underbrace{\frac{\partial}{\partial s} c_s}_{X}, \dot{c}\Big) = 2 g\left(\frac{D}{dt} X, \dot{c}\right)$$

$$\int_0^1 g\left(\frac{D}{dt} X, \dot{c}\right) dt$$

$$= \int_0^1 \left(\left(\frac{d}{dt} g(X, \dot{c})\right) - g\Big(X, \overbrace{\frac{D}{dt} \dot{c}}^{=0}\Big) \right) dt$$

$$= g(X(b), \dot{c}(b)) - g(X(a), \dot{c}(a)) = 0$$

$$\Rightarrow \quad g\left(\frac{D}{dt}X, \dot{c}\right) = 0$$

$\square$

$$\frac{D}{dt}X \parallel \dot{c}$$

In the following let (M, g) be any semi-Riemannian manifold.

Suppose that $c_\bullet : [0,1] \times (-\epsilon, \epsilon) \to M$, $(t, s) \mapsto c_s(t)$ be a variation of $c = c_0$. Recall: The variation vector field of this variation is defined as

$$X(t) := \frac{\partial}{\partial s}\Big|_{s=0} c_s(t).$$

If $c_\bullet$ is a geodesic variation, i. e., if c_s is a geodesic for any $s \in (-\epsilon, \epsilon)$, then the variation vector field J satisfies – generalizing [12, Proposition 3.5.7] in an obvious way:

$$\begin{aligned}
\frac{\nabla^2}{dt^2} J &= \frac{\nabla^2}{dt^2} \frac{\partial}{\partial s}\Big|_{s=0} c_s(t) \\
&= \frac{\nabla}{dt} \frac{\nabla}{ds}\Big|_{s=0} \frac{\partial}{\partial t} c_s(t) \\
&= \frac{\nabla}{dt} \frac{\nabla}{ds}\Big|_{s=0} \dot{c}_s(t) \\
&\overset{(*)}{=} \frac{\nabla}{ds}\Big|_{s=0} \underbrace{\frac{\nabla}{dt} \dot{c}_s(t)}_{=0} + R(\dot{c}, J)\dot{c} \\
&= -R(J, \dot{c})\dot{c}
\end{aligned}$$

At $(*)$ we used Exercise Sheet 4, Exercise 3 c).

Definition 3.2.31. *Let c be a (piecewise C^1-curve. The $J \in \Gamma(c^* TM)$ is called a Jacobi field if*

$$\frac{\nabla^2}{dt^2} J + R(J, \dot{c})\dot{c} = 0.$$

O.D.E. linear of 2nd order

Picard-Lind.

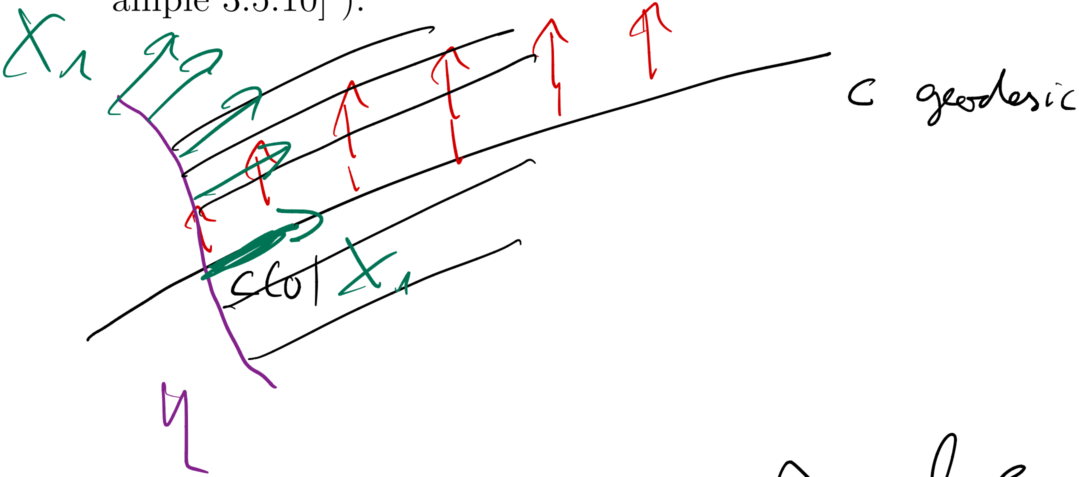
Solution: $J(0)$, $\frac{\nabla}{dt} J(0)$

Lemma 3.2.32. *Let c be a geodesic. For $J \in \Gamma(c^*TM)$ the following are equivalent*

(1) *J is a Jacobi field.*

(2) *there is a geodesic variation $c_\bullet$ of c with variational vector field J.*

Proof:
"(2)$\Longrightarrow$(1)": see above.
"(1)$\Longrightarrow$(2)": (Idea explained in the Riemannian case in [12, after Example 3.5.10]).



Let $c:[a,b]\to M$ be a geod., J a Jacobi field along c. Wlog $0\in[a,b]$.
Choose a curve $\eta:[0,\delta]\ni s\to M$ with $\eta(0)=c(0)$, $\dot\eta(0)=J(0)$. Let $X_1, X_2\in\Gamma(\eta^*TM)$ with $X_1(0)=\dot c(0)$, $X_2(0)=\frac{D}{dt}J|_{t=0}$, and $\frac{D}{ds}X_i=0$.

Define $X(s)=X_1(s)+sX_2(s)$.

$$c_s(t) := \exp_{\eta(s)}(t X(s)).$$

geodesic variation of $c = c_0$

$$Y := \frac{\partial c_s}{\partial s}\Big|_{s=0} \text{ is a Jacobi-field}$$

along c

We will $Y(0) = \mathcal{Y}(0)$ and

$$\frac{D}{dt} Y\Big|_{t=0} = \frac{D}{dt}\mathcal{Y}(0)\Big|_{t=0} \quad (\Rightarrow Y = \mathcal{Y}).$$

$$Y(0) = \frac{\partial c_s}{\partial s}\Big|_{\substack{s=0\\t=0}} = \frac{\partial}{\partial s}\eta(s)\Big|_{s=0} = \mathcal{Y}(0)$$

$$\frac{D}{dt} Y(0)\Big|_{t=0} = \frac{D}{dt}\frac{\partial c_s}{\partial s}\Big|_{\substack{t=0\\s=0}} = \frac{D}{ds}\frac{\partial c_s}{\partial t}\Big|_{\substack{t=0\\s=0}} = \frac{D}{ds}X(s)\Big|_{s=0}$$

$$= X_2(0) + 0 = \frac{D}{dt}\mathcal{Y}\Big|_{t=0}. \quad \square$$