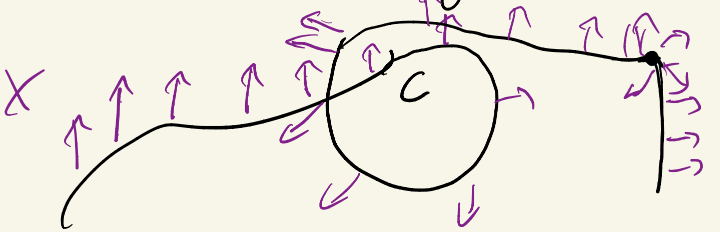
2.2 Curve deformation

X vector field along a curve $c(t)$



$X \in \Gamma(c^* TM)$

$$\frac{\nabla}{dt} X := \nabla_{\frac{\partial}{\partial t}} X$$

$$\boxed{\begin{array}{l} c \text{ geod.} \Leftrightarrow \\ \frac{\nabla}{dt} \dot{c} \equiv 0 \end{array}}$$

<u>Def 2.27</u> A piecewise C^2-curve curve $c : [a,b] \to M$ is a <u>pregeodesic</u> if there is a function $\alpha : [a,b] \to \mathbb{R}$ with

$$\frac{\nabla}{dt} \dot{c}(t) = \alpha(t)\, \dot{c}(t) \qquad \forall t \in [a,b]$$

if c is C^2 around t.

Put $c = \tilde{c} \circ \varphi$ $\qquad \tilde{c} = c \circ \varphi^{-1}$

$\varphi : [a,b] \to [c,d]$

(orient.-preserving) diffeom.

then $$\frac{\nabla}{dt}\dot{c} = \frac{\nabla}{dt}\left(\dot{\varphi}\cdot(\dot{\tilde{c}}\circ\varphi)\right)$$

$$= \ddot{\varphi}\cdot(\dot{\tilde{c}}\circ\varphi) + (\dot{\varphi})^2\Big(\underbrace{\left(\frac{\nabla}{dt}\dot{\tilde{c}}\right)\circ\varphi}_{\tilde{\lambda}(t)\,\dot{\tilde{c}}\circ\varphi(t)}\Big)$$

Thus c geodesic $\Longrightarrow$

c pregeodesic $\Longleftrightarrow$ $\tilde{c}$ pregeodesic

$\Longleftarrow$ $\tilde{c}$ geodesic

Vice versa every pregeodesic can be reparametrized to a geodesic

If c is a pregeodesic with $\hat{c}$ C^2

$$\frac{D}{dt}\dot{c} = \alpha(t)\,\dot{c}(t)$$

Then $\hat{c}$ is a geodesic

$$\Longleftrightarrow \frac{D}{dt}\dot{\hat{c}} = 0$$

$$\Longleftrightarrow \alpha(t)\,\dot{c}(t)$$

$$= \ddot{\varphi}(t)\cdot\left(\dot{\hat{c}}\circ\varphi(t)\right)$$

$$\Longleftrightarrow \alpha(t)\,\dot{\varphi}(t)$$

$$= \ddot{\varphi}(t)$$

$$\Longleftrightarrow \alpha(t) = \frac{\ddot{\varphi}(t)}{\dot{\varphi}(t)} = \frac{d}{dt}\log|\dot{\varphi}(t)|$$

$$\varphi(t) = \pm \int_a^t \exp\left(\int_0^\tau \alpha(s)\,ds + c_1 \right) d\tau$$

$$\Leftrightarrow \quad \varphi(t) = \left(\underset{\shortparallel}{\overset{\pm c_1}{C}} \int_a^t e^{\int_0^\tau \alpha(s)\,ds} \right) + c_2$$

$$\pm \exp c_1$$

<u>Compact-open topology:</u>

X, Y Hausdorff spaces

Compact open top. is a topology on $C(X,Y)$.

$= \{ f : X \to Y \text{ continuous} \}$

(Definition: Exercise sheet)

$X = [a,b]$, $M = Y$ mfd., connected, $\partial M = \emptyset$

We equip $C([a,b], M)$

with compact-open topology

Consider a Riemannian metric given on M.

$\rightsquigarrow d(x,y) := \inf\{L[c] \mid c$ is a curve from x to $y\}$

(M,d) metric space

Equip $C([a,b], M)$ with the supremum metric, i.e.

i.e. $c_1, c_2 \in C([a,b], M)$

$$d_\infty(c_1, c_2) := \sup_{t\in[a,b]}^{\max} \left\{ d(c_1(t), c_2(t)) \right\}$$

$\in [0,\infty)$

$(C([a,b], M), d_\infty)$ metric space

<u>Exercise</u> The induced topology on $C([a,b], M)$ is the compact open topology.

Thus the topology on $C([a,b], M)$

does not depend on the choice of g_{Riem}.

$c_i \to c$

$\Longleftrightarrow$ $c_i(t) \longrightarrow c(t)$ uniformly in t

$\overset{(def)}{\Longleftrightarrow}$ $\forall \varepsilon > 0\ \exists i_0\ \forall i \geq i_0\ \forall t \in [a,b]$

$$d(c_i(t), c(t)) < \varepsilon$$

compact-open topology = topology of uniform convergence.

Warning: Does not hold for $C(I, M)$, I non-compact.

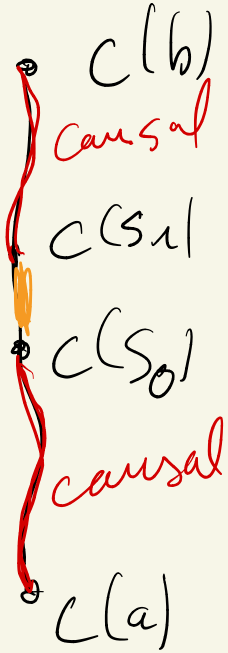
Lemma 2.29 Let $c : [a,b] \to M$ be a piecewise C^1 causal curve which is not a lightlike pregeodesic.
In each neighborhood of c (in the compact-open topology) we find a timelike smooth curve with the same start and end point.

$p < q \qquad p \ll q$

Proof 1.) In Cor 2.17a)
we showed that the conditions
of the lemma imply that we
have a smooth causal curve $\tilde{c}$
from $c(a)$ to $c(b)$.
Looking at the proof of this
Corollary we see that $\tilde{c}$ can
be chosen in a given
(compact-open) neighborhood
of c.
Thus w.l.o.g. c is smooth.

2.) Suppose $\dot{c}(t_0)$ is timelike for some $t_0 \in [a,b]$.

$\Rightarrow \exists\, s_0, s_1$ with $a \leq s_1 < s_2 \leq b$ such $\forall t \in [s_0, s_1]$: $\dot{c}(t)$ is timelike.



Idea Does not yet yield a proof!

$$c(a) \ll c(b)$$

$\Uparrow$ Prop 2.7

$$c(a) \leq c(s_0) \ll c(s_1) \leq c(b)$$

In the proof of Prop 2.7 we constructed a timelike curve with the same start and end point by a deformation argument. Thus the proof even gives us a timelike curve with same start and end point in the given nbhd of c.

3.) Let c be light like, but not a pregeodesic.

Wlog $[a,b]=[0,1]$.

$$\Rightarrow g(\dot c(t), \dot c(t)) = 0 \quad \forall t\in[0,1]$$

$$0 = \frac{d}{dt} g(\dot c(t), \dot c(t))$$

$$= 2g\left(\frac{D}{dt}\dot c(t), \dot c(t)\right)$$

$$\Rightarrow \frac{D}{dt}\dot c(t) = \dot c(t)^{\perp}$$

If $v \neq 0$ in a Minkowski space, if v is lightlike, then $v^{\perp} = \mathbb{R}v \oplus E$

E spacelike, i.e.

$g|_{E\times E}$ is pos. def.

$\dim E = \dim M - 2$.

$$\frac{D}{dt}\dot{c}(t) \in \mathbb{R}\dot{c}(t) \oplus \underbrace{E_{c(t)}}_{\subset T_{c(t)}M}$$

spacelike, codim 2

As c is not a pregeodesic, there is a $t_0 \in [0,1]$, s.that

$$\frac{D}{dt}\dot{c}(t_0) \notin \mathbb{R}\dot{c}(t_0)$$

$$\Rightarrow g\left(\frac{D}{dt}\Big|_{t=t_0}\dot{c}, \frac{D}{dt}\Big|_{t=t_0}\dot{c}\right) > 0$$

$$0 = \frac{d}{dt}\, g\left(\frac{D}{dt}\dot{c}(t), \dot{c}(t)\right)$$

negative at $t = t_0$

$$= g\left(\frac{D}{dt}\frac{D}{dt}\dot{c}(t), \dot{c}(t)\right)$$

$$+ g\left(\frac{D}{dt}\dot{c}(t), \frac{D}{dt}\dot{c}(t)\right)$$