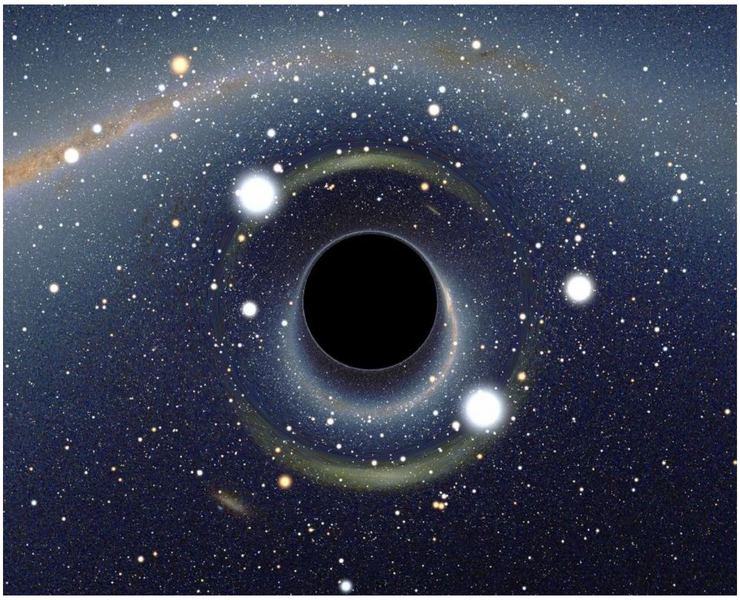
# Differential Geometry II
# Lorentzian Geometry
## Lecture Notes





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Presentation Version

Version of May 30, 2021

$\tau : M \times M \to \mathbb{R}$ denotes the time-separation.
$\geq 0$

$$\tau(p,q) := \begin{cases} \sup\left\{ \mathcal{L}[c] \;\middle|\; \begin{array}{l} c \text{ is a causal future di-} \\ \text{rected curve from } p \text{ to } q \end{array} \right\} & \text{if } p < q \\ 0 & \text{if } p \not< q \end{cases}$$

**Proposition 3.2.26.**

(1) $\tau(p,q) > 0 \quad \Longleftrightarrow \quad p \ll q$,

(2) *If* $p \leq q$ *and* $q \leq r$*, then* $\tau(p,q) + \tau(q,r) \leq \tau(p,r)$ *(inverse triangle inequality),*

(3) *The function* $\tau : M \times M \to \mathbb{R}$ *is lower semi-continuous.*

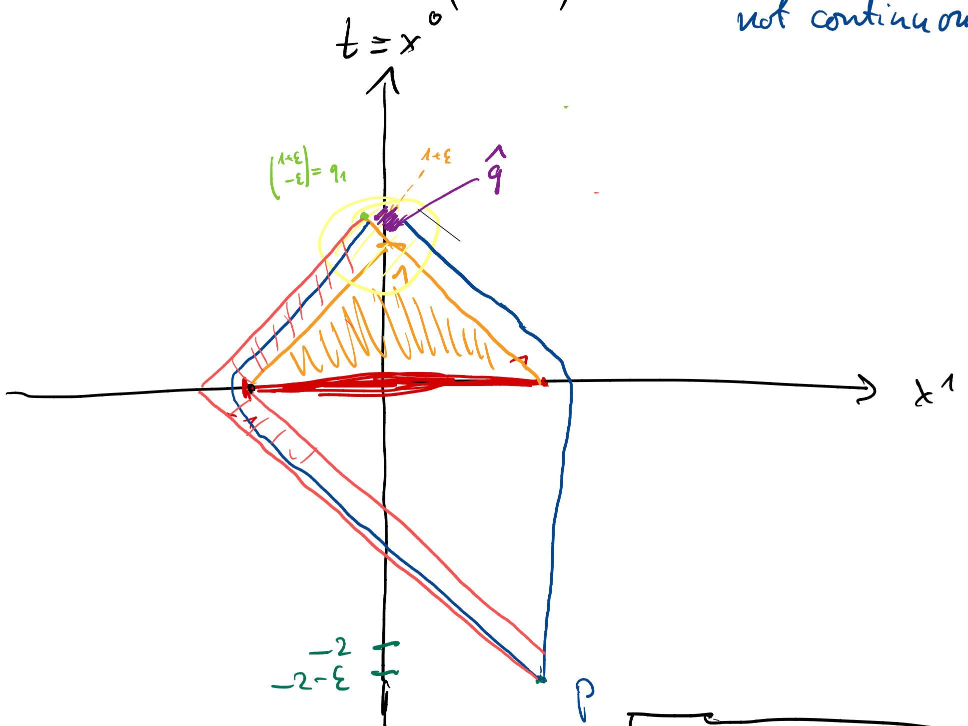
$\tau : M \times M \to \mathbb{R}$ denotes the time-separation.

$$\tau(p,q) := \begin{cases} \sup\left\{\mathcal{L}[c] \;\middle|\; \begin{array}{l} c \text{ is a causal future di-} \\ \text{rected curve from } p \text{ to } q \end{array}\right\} & \text{if } p < q \\ 0 & \text{if } p \not< q \end{cases}$$

**Remark.** In general $\tau : M \times M \to \mathbb{R}$ is not continuous, as we will see in the follwoing example.

$M = \mathbb{R}^{1,1} \setminus \big(\{0\} \times [-1,1]\big)$, $p = \begin{pmatrix} -2-\varepsilon \\ 1 \end{pmatrix}$

$q \longmapsto \tau(p,q)$
not continuous in




$$\tau(p,q) = \sup_{\tilde{q}\,\varepsilon \dots} \tau(p,\tilde{q}) + \tau(\tilde{q},q_1) \leq \sqrt{\left|\left\langle\!\left\langle \begin{pmatrix} 2+\varepsilon \\ 2 \end{pmatrix}, \begin{pmatrix} 2+\varepsilon \\ 2 \end{pmatrix} \right\rangle\!\right\rangle\right|} + \sqrt{\left|\left\langle\!\left\langle \begin{pmatrix} 1+\varepsilon \\ 1-\varepsilon \end{pmatrix}, \begin{pmatrix} 1+\varepsilon \\ 1-\varepsilon \end{pmatrix} \right\rangle\!\right\rangle\right|} \leq \otimes$$

on the ~~line~~ segment from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1+\varepsilon \\ -\varepsilon \end{pmatrix}$.

$\tilde{q}$ any point on the segm. from $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ -1-\varepsilon \end{pmatrix}$

$$\circledast \sqrt{|-(2+\varepsilon)^2+2|} + \sqrt{|-(1+\varepsilon)^2+(1-\varepsilon^2)|}$$

$$= \sqrt{4\varepsilon+\varepsilon^2} + \sqrt{4\varepsilon} \le 5\sqrt{\varepsilon}$$

As $\varepsilon \to 0$ this goes to $0$

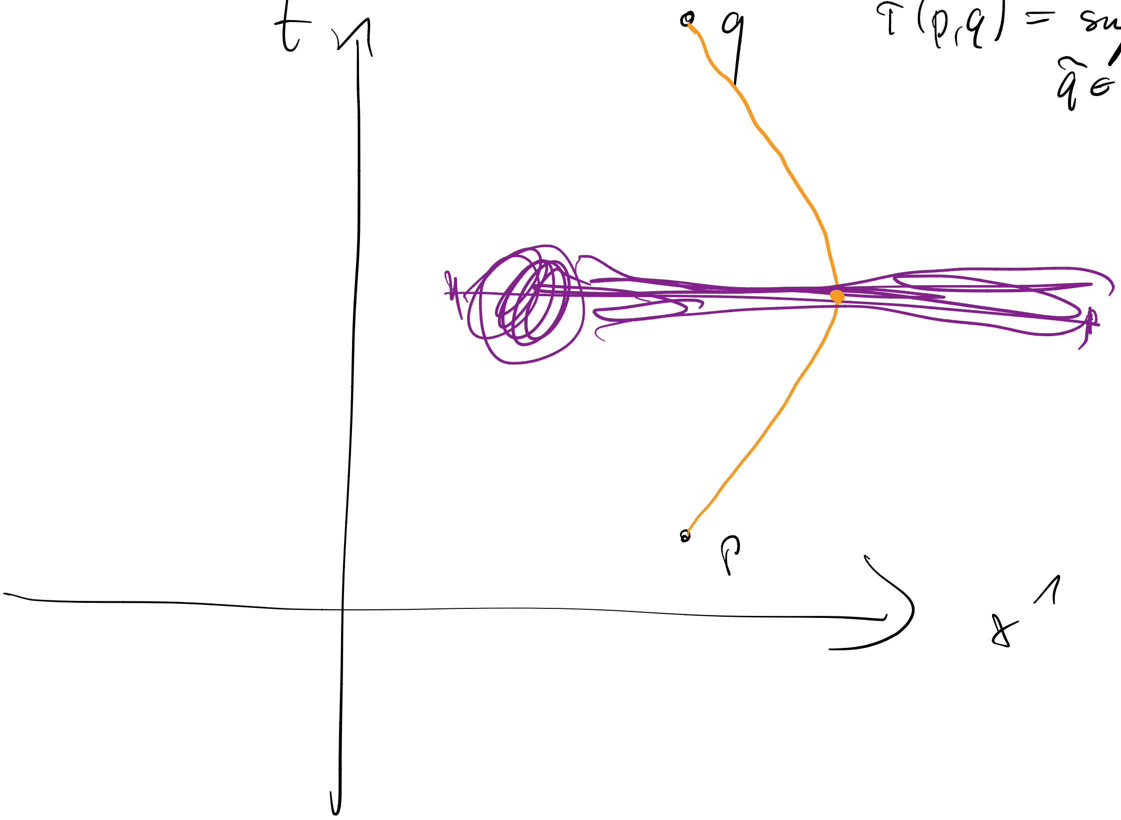
For any point $\hat{q}$ in the violett region there is a f.d. causal curve $c$ from $p$ to $\hat{q}$ such $L[c] \ge 1-\varepsilon$.

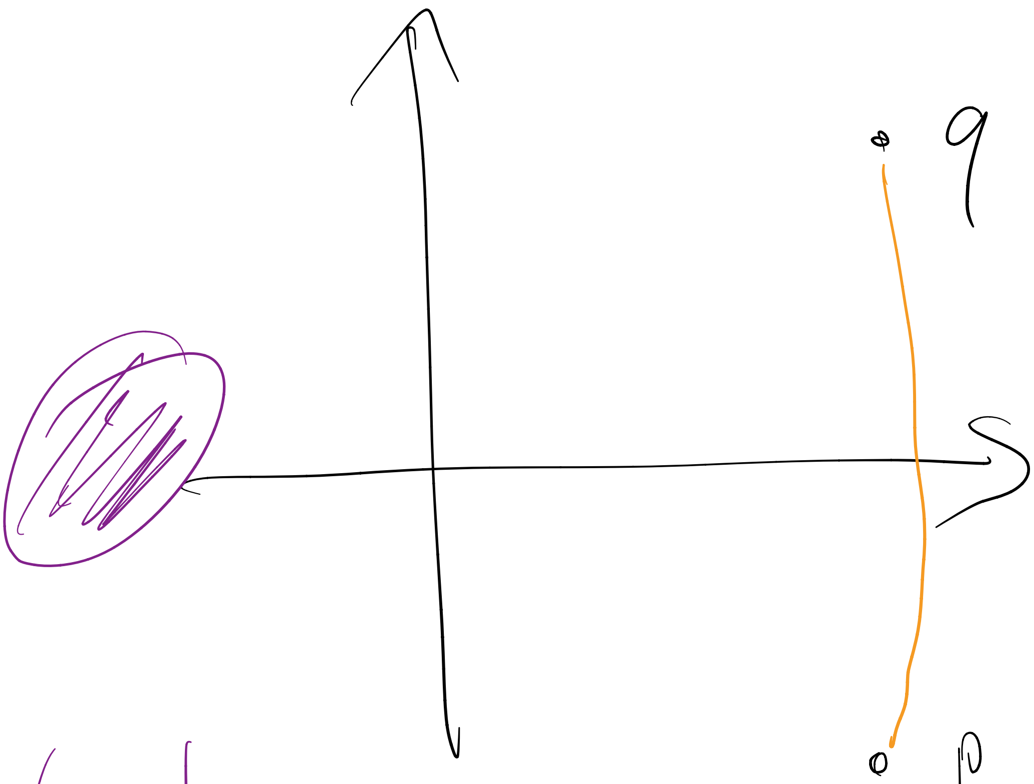
$$\tau(p,\hat{q}) \ge 1-\varepsilon.$$

Discontinuity at segment from $\begin{pmatrix}1\\0\end{pmatrix}$ to $\begin{pmatrix}1+\varepsilon\\-\varepsilon\end{pmatrix}$.

### 3.2.2 Curve deformation

$$\tau(p,q) = \sup_{\tilde{q} \in \,\cdot\,} \{\tau(p,\tilde{q}) + \tau(\tilde{q},q)\}$$




$\tau(p,q) > 0$

$\tau(p,\tilde{q}) = 0 \quad \tau(\tilde{q},q) = 0$