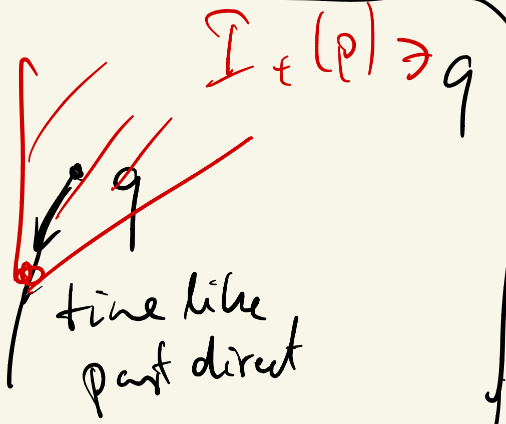
Prop 2.18

Any compact Lorentzian manifold M carries a smooth periodic timelike curve $c : \mathbb{R} \to M$, i.e. $L > 0$ with $c(t+L) = c(t)$ $\forall t \in \mathbb{R}$ and $\dot{c}(t)$ timelike $\forall t \in \mathbb{R}$.

Prf: 1) Assume that M is time-oriented.

$\big(\underbrace{I_f(p)}_{\text{open}}\big)_{p\in M}$ is an open cover of M. $\exists\, p_1,\dots,p_N \in M$



M cpct $\Longrightarrow$ $M = I_f(p_1) \cup I_f(p_2) \cup \dots \cup I(p_N)$

We can achieve that

$$I_f(p_i) \not\subseteq I_f(p_j) \text{ for } i \neq j$$

$p_1 \in M \Longrightarrow \exists\, j = 1,\dots,N$ with

$p_1 \in I_+(p_j)$

If $j \neq 1$ then $\underset{\substack{\cup \\ q}}{I_+(p_1)} \subset I_+(p_j)$

$\Rightarrow p_j \ll p_1 \ll q$

$\Rightarrow p_j \ll q$

$\Rightarrow q \in I_+(p_j)$

↯

Thus $j = 1$. Then $p_1 \in I_+(p_1)$

$p_1 \ll p_1$. Hence there is

a piecewise C^1-curve, timelike

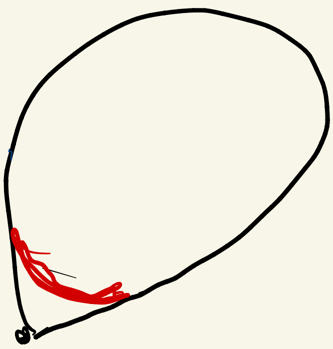
future directed $c : [0, 1] \to M$

$c(0) = p_1$ and $c(1) = p_1$.

$c(t+k) := c(t) \quad \forall k \in \mathbb{Z}, t \in [0,1]$

$c : \mathbb{R} \to M$

One can use the techniques from Cor 2.17a) to argue that there a curve $\hat{c}$ with the same properties as c, but additionally smooth.



[but not necessarily running through p_1]

2.) For arbitrary M,

there is a time-oriented $\hat{M}$ with

$$\pi : \hat{M} \xrightarrow{2:1} M \quad \Rightarrow \hat{M} \text{ cpt}$$

smooth local diffeo and isometry

(Ex 2 on sheet no. 6)

By 1) we

$$\begin{tikzcd} & \hat{M} \arrow[d, "\pi"] \\ \mathbb{R} \arrow[ur, "\hat{c}"] \arrow[r, "c := \pi \circ \hat{c}"'] & M \end{tikzcd}$$

$\square$

"2:1": surjective

$$\#(\pi^{-1}(p)) = 2 \quad \forall p \in M$$

Again let M a connected, time-oriented Lorentzian mfd.

Def 2.19 M satisfies the

* chronology condition

$:\Leftrightarrow$ there is no smooth closed timelike curve in M

Cor 2.17 a)

$\Longleftrightarrow$ there is no piecewise C^1-curve in M that is closed, timelike and future directed.

sim. to Cor 2.17

$\Longleftrightarrow$ there is no smooth periodic timelike curve in M.

* causality condition

:<=> there is no smooth

closed / closed, piecewise C^1,

future directed / smooth periodic

causal curve in M.

* strong causality condition

:<=> for any $p \in M$ and any

neighborhood U of p there is

a nbhd $V \subset U$ of p such that

every future directed causal piecewise C^1-curve $\gamma : [a,b] \to M$ with $\gamma(a), \gamma(b) \in V$ satisfies $\gamma([a,b]) \subset U$.

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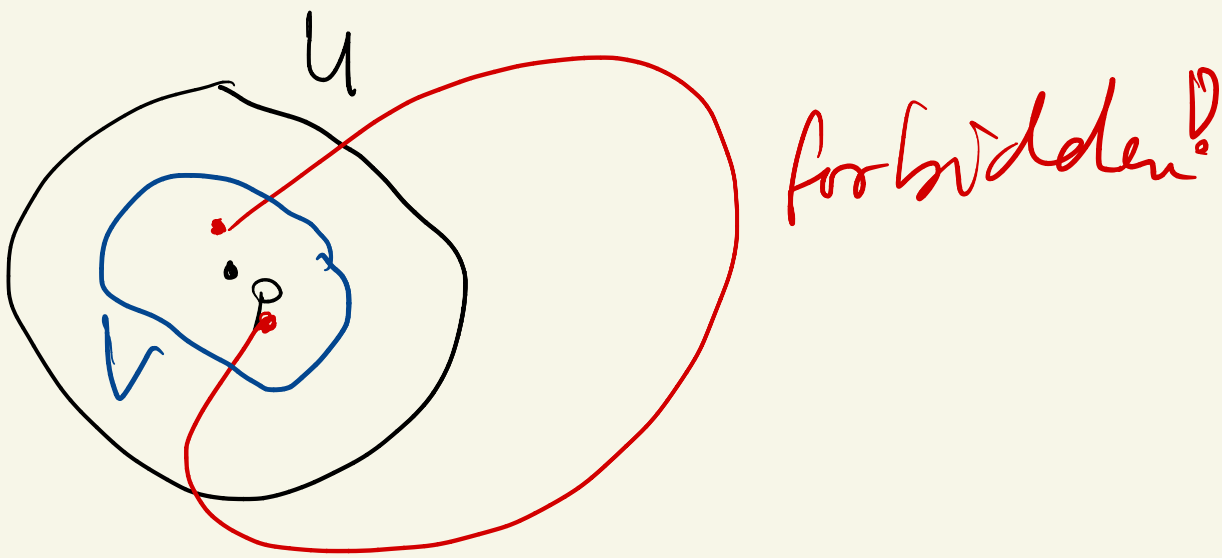
strong causality condition

$\Rightarrow$ causality condition

$\Rightarrow$ chronology condition



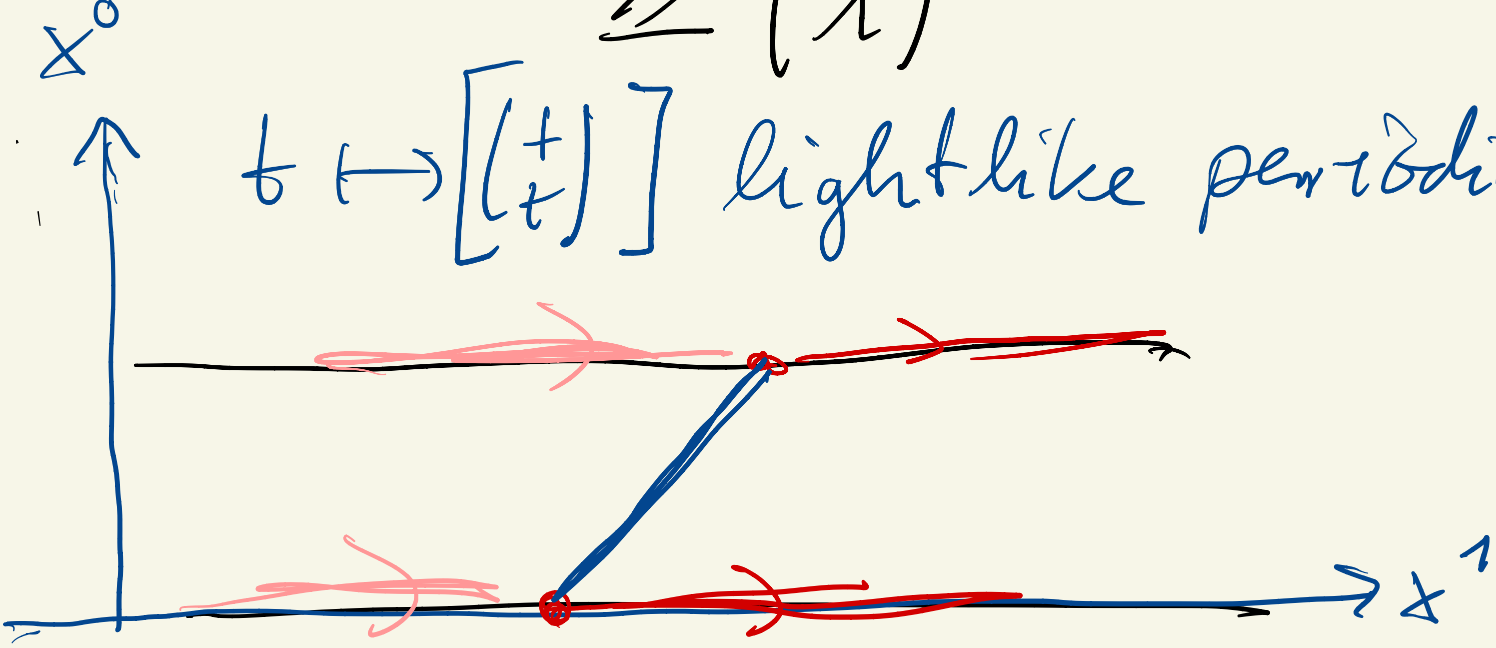
Examples 1) M cpct

chronology violated

(= not satisfied = does not hold)

2) $M = \mathbb{R}^{1,1} / \mathbb{Z} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$t \longmapsto \left[\begin{pmatrix} t \\ t \end{pmatrix} \right]$ lightlike periodic



smooth curve ($\Rightarrow$ causal)

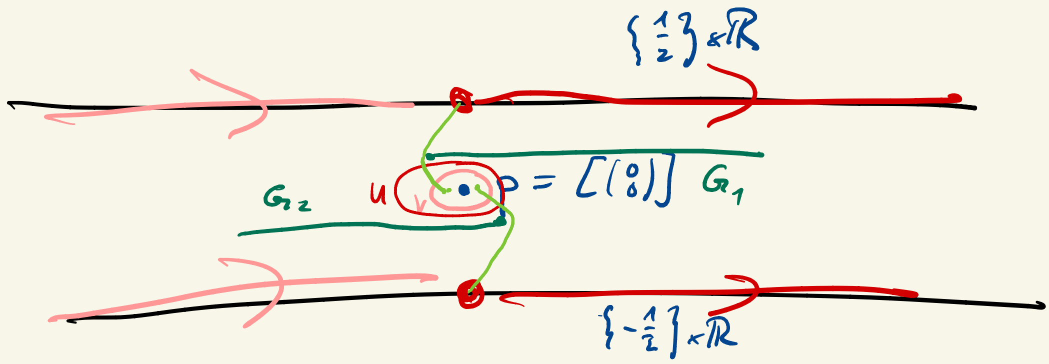
$\Rightarrow$ causality condition violated

If $c : \mathbb{R} \to M$ is periodic, say $c(t+L) = c(t)$, then $c(L) = c(t) + a\begin{pmatrix}1\\1\end{pmatrix}$, $a \in \mathbb{Z}$. If c is causal & future directed then $c(t) = (c^0(t), c^1(t))$, $\dot{c}^0(t) \geq |\dot{c}^1(t)|$, then $\dot{c}^0(t) = |\dot{c}^1(t)|$.

No closed timelike curves in M

$\Rightarrow$ chronology condition satisfied.

3) $$M = \left(\mathbb{R}^{1,1} \Big/ \mathbb{Z} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \setminus (G_1 \cup G_2)$$



$$G_1 := \left\{ \left[\begin{pmatrix} \frac{1}{8} \\ s \end{pmatrix} \right] \;\middle|\; s \geq -\frac{1}{8} \right\}$$

$$G_2 := \left\{ \left[\begin{pmatrix} -\frac{1}{8} \\ s \end{pmatrix} \right] \;\middle|\; s \leq \frac{1}{8} \right\}$$

strong causality is not satisfied

satisfies causality.

<u>Def 2.24</u> $c : [a,b] \to M$

length $\mathcal{L}[c] := \int_a^b \sqrt{|g(\dot{c}(\tau), \dot{c}(\tau))|}\, d\tau$

If c is causal, then $\mathcal{L}[c]$ is also called the proper time of c.

For $p, q \in M$ define the time-difference

$$\tau(p,q) := \begin{cases} \sup \left\{ \mathcal{L}[c] \;\middle|\; \begin{array}{l} c \text{ is causal} \\ \text{future directed} \\ \text{from } p \text{ to } q \end{array} \right\} & \text{if } p < q \\ 0 & \text{if } p \not< q \end{cases}$$

$\in [0, \infty]$

__Example__ 1) $M = \mathbb{R}^{n,1}$

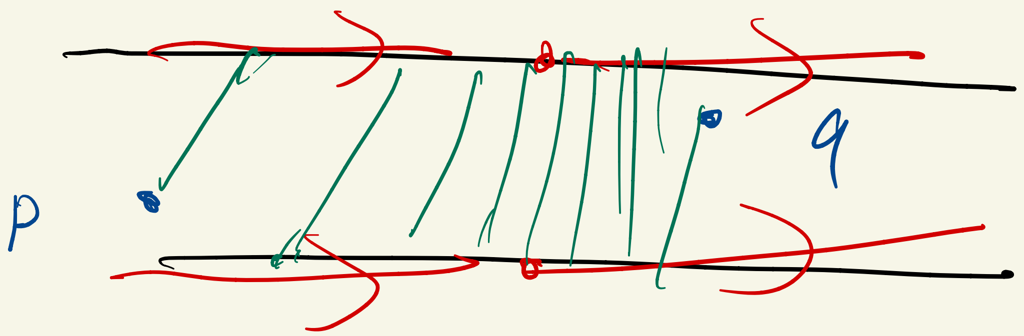
[see Ex 3 of sheet no. 1]

$$\tau(p,q) = \begin{cases} \sqrt{|\langle\langle q-p, q-p \rangle\rangle|} & \text{if } p < q \\ 0 & \end{cases}$$

$\tau : M \times M \to \mathbb{R}$ continuous.

2) $M = \mathbb{R}^{1,1} / \mathbb{Z} \cdot \binom{1}{0}$ $\forall p,q \in M$

$\tau(p,q) = \infty$



Prop 2.26

1) $\tau(p,q) > 0 \iff p \ll q$

2) $p \leq q$ and $q \leq r$, then

$$\tau(p,q) + \tau(q,r) \leq \tau(p,r)$$

(inverse triangle inequality)

3) $\tau : M \times M \to \mathbb{R}$ is lower semicontinuous.

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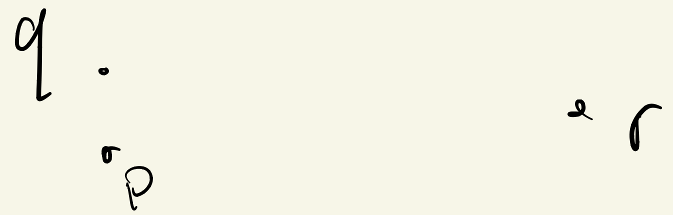
in $\mathbb{R}^{1,1}$



$\tau(p,q) > 0$, $\tau(q,r) = 0$
$\tau(p,r) = 0$

Pf: 1) "$\Leftarrow$" $p \ll q$

$\Rightarrow \exists$ timelike future-directed curve γ from p to q.

$\Rightarrow \tau(p,q) \geq L[\gamma] > 0$.

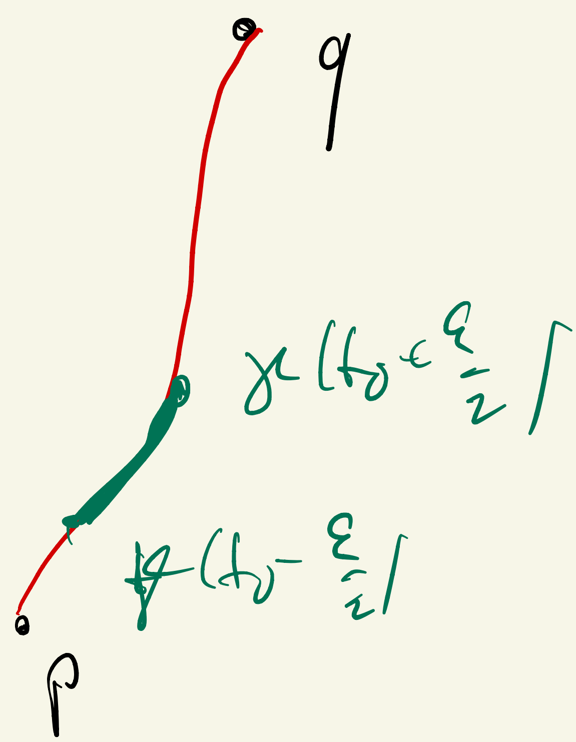
"$\Rightarrow$" $\tau(p,q) > 0 \Rightarrow \exists$ causal future directed curve γ from p to q, $\gamma : [a,b] \to M$, $b > a$ with $L[\gamma] > 0$. Assume for simplicity C^1.

$\Rightarrow g(\dot\gamma(t), \dot\gamma(t)) < 0$ ~~for~~ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \subset [a,b]$

$$\left[t_0 - \frac{\varepsilon}{2}, t_0 + \frac{\varepsilon}{2}\right] \subset (t_0 - \varepsilon, t_0 + \varepsilon)$$

$$\underset{\gamma(a)}{\underset{\shortparallel}{p}} \le \gamma\left(t_0 - \frac{\varepsilon}{2}\right) < \gamma\left(t_0 + \frac{\varepsilon}{2}\right) \le q = \gamma(b)$$

$$\overset{\text{Prop } 2.7}{\Longrightarrow} \quad p \ll q$$



2) Let c_1 be a causal future directed curve from p to q

Let c_2 " " " " " " " q to r

$c_1 * c_2$ curve from p to r

Assume $\mathcal{L}[c_1] \geq \tau(p,q) - \varepsilon$ and $\mathcal{L}[c_2] \geq \tau(q,r) - \varepsilon$ (if both $\tau(p,q)$ and $\tau(q,r)$ are finite, otherwise assume $\mathcal{L}[c_1] \geq L$ or $\mathcal{L}[c_2] \geq L$)

$$\tau(p,r) \geq L[c_1 * c_2]$$

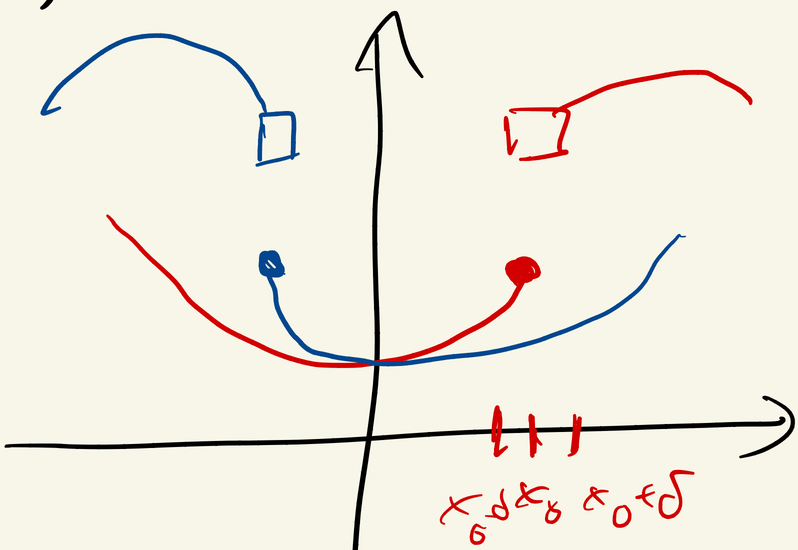
$$= L[c_1] + L[c_2]$$

$$\geq \begin{cases} \tau(p,q) - \varepsilon + \tau(q,r) - \varepsilon & \text{if } \tau(p,q) < \infty \text{ and } \tau(q,r) < \infty \\ \geq L & \text{otherwise} \end{cases}$$

Take $\varepsilon > 0$ small or L large

$\Longrightarrow$ statement of 2)

3) Recall $f: \mathbb{R} \to \mathbb{R}$



f is lower semi-continuous

$\forall x_0 \in \mathbb{R}, \forall \varepsilon > 0 \; \exists \delta > 0$

$$f([x_0-\delta, x_0+\delta]) \subset [f(x_0)-\varepsilon, \infty)$$

Generalizes in a obvious for fctions $f: M \times M \to \mathbb{R}$

~~Let~~ Case 1) $\tau(p,q) = 0$: ✓

Use (1)

Case 2) $0 < \tau(p,q) < \infty$ (~~Use (2)~~)

Let $c : [0,1] \to M$ be a

causal, ~~timelike~~ , future-directed

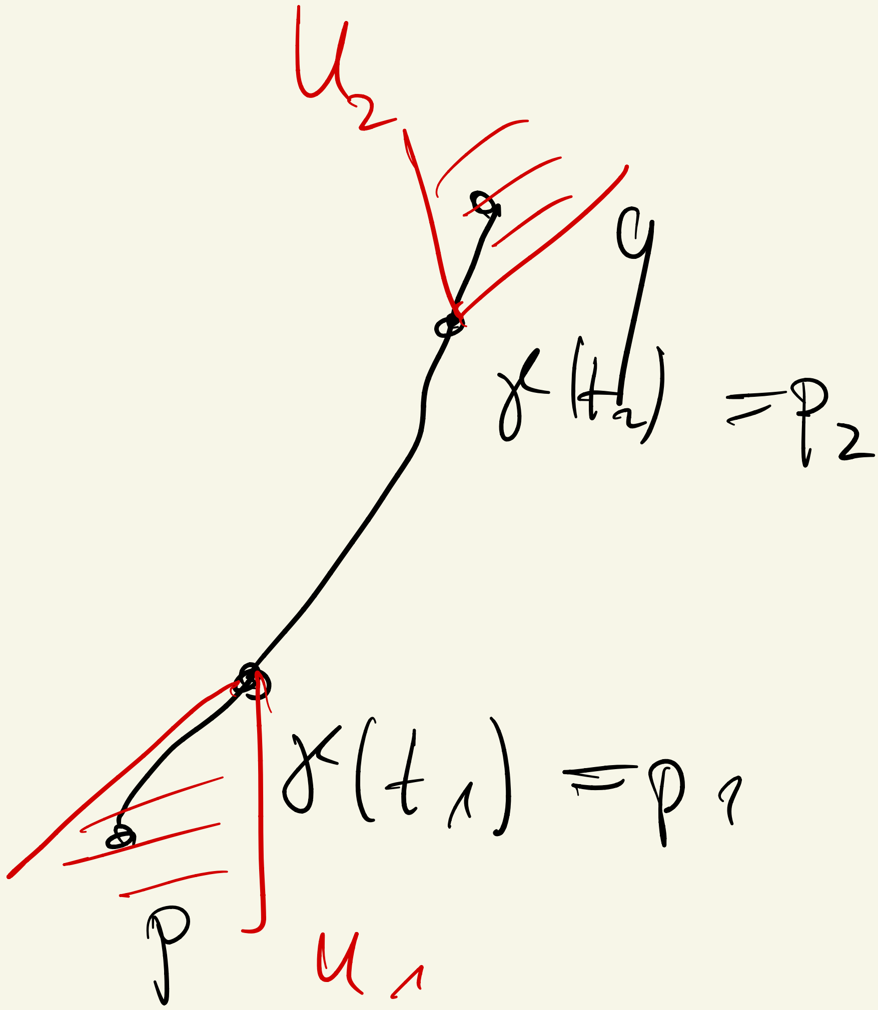
curve with $\mathcal{L}[c] \geq \tau(p,q) - \frac{\varepsilon}{2}$

$0 < \varepsilon < \frac{\tau(p,q)}{2}$

Choose $0 < t_1 < t_2 < 1$ s.th.

$0 < \mathcal{L}\left(c|_{[0,t_1]}\right) < \frac{\varepsilon}{4}$ and

and $0 < L[c_{|[t_2,1]}] < \frac{\varepsilon}{4}$



$p \ll \gamma(t_1) \ll \gamma(t_2) \ll q$

$p \in \underbrace{I_-(p_1)}_{=: U_1} \qquad q \in \underbrace{I_+(p_2)}_{=: U_2}$

$U_1 \times U_2$ is an open nbhd of (p,q) in $M \times M$.

Let $q' \in U_2$, $p' \in U_1$

$$\tau(p',q') \geq \overbrace{\tau(p',p_1)}^{\geq 0} + \underbrace{\tau(p_1,p_2)}_{} + \underbrace{\tau(p_2,q')}_{\geq 0} \geq \tau(p,q) - \varepsilon$$

$$\geq \tau(p,q) - \varepsilon$$

$$\tau(p',q') \geq \tau(p,q) - \varepsilon$$

$\Longrightarrow$ Case 2

Case 3: $\tau(p,q) = \infty$ similar $\square$