Proof: Consider (+)-case. (-)-case similar

1.) $\subset$ $I_+^{\Omega}(p) \subset \exp_p\left(I_+^{T_pM}(0) \cap \Omega'\right)$

$q \in I_+^{\Omega}(p)$, i.e. exist a piecewise C^1-curve $c : [0,1] \to \Omega$ timelike, future directed with $c(0) = p$ and $c(1) = q$.

Apply Lemma 2.11 $\gamma := \exp_p^{-1} \circ c : [0,1] \to \Omega'$,

$\overset{L\,2.11}{\Longrightarrow}$ $\gamma : (0,1] \to I_+^{T_pM}(0)$ $\Rightarrow$ $q = \exp_p(\gamma(1))$
$\in \exp_p\left(I_+^{T_pM}(0) \cap \Omega'\right)$

2.) $\supset$ Let $x \in I_+(0) \cap \Omega'$

$\gamma(t) := tx$ $\quad \gamma(0) = 0,\ \gamma(1) = x$

$c := \exp_p \circ \gamma$ is a geodesic in M

with $\dot{c}(0) = x$ $\Rightarrow$ $\dot{c}(t)$ timelike, future-directed.

$c(t) \in \Omega$

$c(1) = \exp_p(x) \in I_+^{\Omega}(p)$

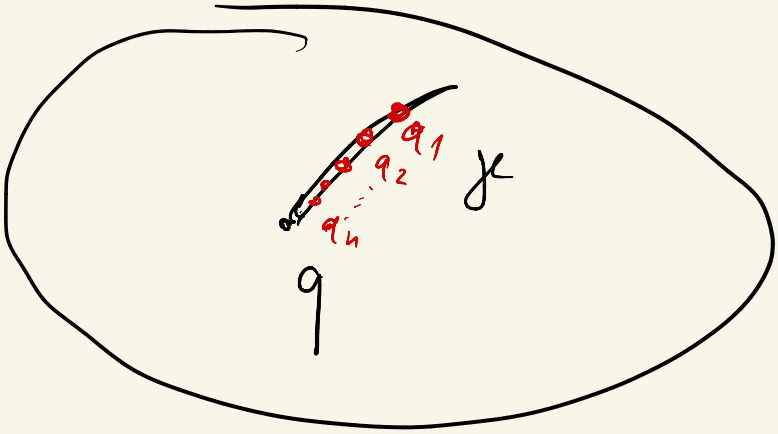
$c(0) = p$

$$I_+^{\Omega}(p) = \exp_p\left(I_+^{T_pM}(p) \cap \Omega'\right)$$

3.) $\mathcal{J}_-$?

complete analogous in 2)

4.) Let $q \in \mathcal{J}_+^\Omega(p)$.



γ starts in q

$\gamma(0) = q$

γ time like

$$q_i := \gamma\left(\frac{1}{i}\right) \in I_+^\Omega(q)$$

$$q_i \to q$$

Prop 2.7 $\Rightarrow$

$\underset{\text{in } \Omega}{p \leq q}$ and $\underset{\text{in } \Omega}{q \ll q_i}$

$\underset{\text{in } \Omega}{p \ll q_i} \Rightarrow q_i \in I_+^\Omega(p)$

1)

$$\Rightarrow \underbrace{\exp_p(q_i)}_{\to (\exp_p|_{\Omega'})^{-1}(q)} \in I_+^{T_pM} \cap \Omega'$$

$$\Rightarrow \exp_p^{-1}(q) \in \underbrace{\overline{I_+^{T_pM}(0)}}_{= J_+^{T_pM}(0)} \cap \Omega'$$

$$q \in \exp_p\left(\Omega' \cap J_+^{T_pM}(0)\right)$$

$\underline{\Omega}$

<u>Prop 2.14</u> "$\ll$" is an open relation, i.e.

$\{(p,q) \in M \times M \mid p \ll q\}$ is an open subset of $M \times M$

<u>Proof</u>: Assume $p \ll q$.

$\exists\, c : [0,1] \to M$ timelike future-directed

Take $\varepsilon > 0$ such that

$$c([0,\varepsilon]) \subset \exp_{c(t)}(\mathcal{D}_{c(t)})$$

$\forall t \in [0,\varepsilon]$ and

we Ω'_t starshaped

nbhd of 0 in $T_{c(t)}M$

such Cor 2.13

For suff. small t we

have $p \in \Omega :=$

$\exp_{c(t)}(\Omega'_t)$

$p' = c(t)$

$\overset{\text{Cor}}{\Longrightarrow} p \in I_-^{\Omega}(p')$

$I_-^{\Omega}(p')$ is an open nbhd of
p by Cor 2.13.

Do the same at the other end. For sufficiently small t we obtain

$q' := c(1-t)$

$q \in I_+^{\tilde{\Omega}}(q')$ open nbhd of q.

For any $(\hat{p}, \tilde{q}) \in I_-^{\Omega}(p') \times I_+^{\hat{\tilde{\Omega}}}(q')$

$\tilde{p} \ll p' \ll q' \ll \tilde{q}$

$\Rightarrow I_-^{\Omega}(p') \times I_+^{\hat\Omega}(q')$

open nbhd of (p, q) in

$\{(p,q) \in M \times M \mid p \ll q\} =: P$

$\square$

<u>Cor 2.15</u>

$I_{\pm}(p)$ is open

$I_\pm(A) = \bigcup_{p \in A} I_\pm(p)$ open

$\square$

$\{p\} \times I_+(p)$

$= P \cap (\{p\} \times M) \subseteq \{p\} \times M$

Prop 2.17

(1) $I_\pm(A) = \overbrace{J_\pm(A)}^{\circ}$

(2) $J_\pm(A) \subset \overline{I_\pm(A)}$

(3) $\overline{J_\pm(A)} = \overline{I_\pm(A)}$

Proof (1) (a)

$\underbrace{I_\pm(A)}_{\text{open}} \subset J_\pm(A)$

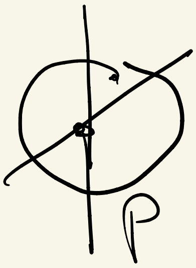
$$\Rightarrow I_\pm(A) \subset \overbrace{J_\pm(A)}^{\circ}$$

(b) $p \in \overbrace{J_+(A)}^{\circ}$

Choose a point $q \in I_-(p)$

such $q \in \overset{\circ}{\overbrace{J_+(A)}} \subset J_+(A)$

$$\Rightarrow \exists\, r \in A \quad r \leq q \ll p$$



$$\Rightarrow r \ll p$$

$$\Rightarrow p \in I_+(r) \subset I_+(A)$$

$$\Rightarrow \overset{\circ}{\overbrace{J_+(A)}} \subset I_+(A)$$

the same holds J_-, I_-.

(2) $J_+(p) \subset \overline{I_+(p)}$:

Let $q \in J_+(p)$ $\nearrow$ $q = p$

$q = p \in \overline{I_+(p)}$

$\hookrightarrow$ $p < q$.

There is a piecewise

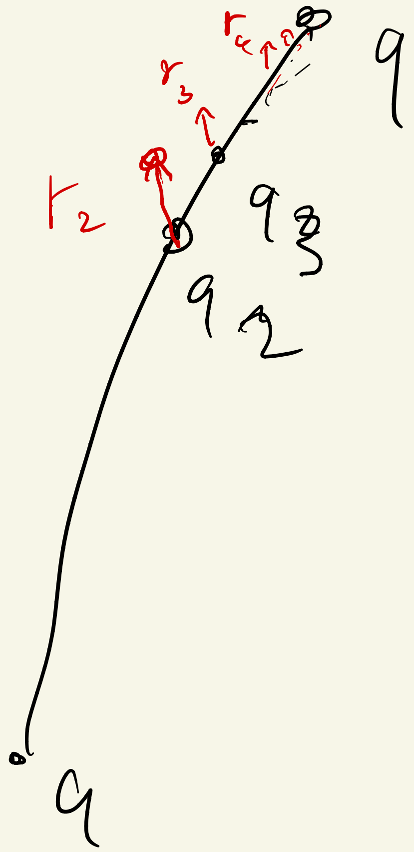
C^1-curve $c : [0, 1] \to M$

causal, future-directed

from $p = c(0)$ to $q = c(1)$.

Set $q_i := c(1 - \frac{1}{i})$ $\quad q_i \to q$

$i > 1$



Choose $r_i >> q_i$

with $r_i \to q$

$$p < q_i << r_i$$

$$\Rightarrow r_i \in I_+(p)$$

$$\Rightarrow q \in \overline{I_+(p)}$$

(b) $$J_+(A) = \bigcup_{p \in A} J_+(p)$$

$$\subset \bigcup_{p \in A} \underbrace{\overline{I_+(p)}}_{\subset \overline{I_+(A)}} \subset \overline{I_+(A)}$$

$$I_+(p) \subset \underset{p \in A}{I_+(A)}$$

(3) $\overline{J_+(A)} \subset \overline{I_+(A)}$ follows from (2)

$$I_{\pm}(A) \subset \mathcal{J}_{\pm}(A)$$

$$\Rightarrow \overline{I_{\pm}(A)} \subset \overline{\mathcal{J}_{\pm}(A)}$$

$\square$

Cor 2.17a)

1) If $p \ll q$, then there is a smooth timelike future directed path from p to q

2) If $p < q$, then there is a smooth causal future directed path from p to q.

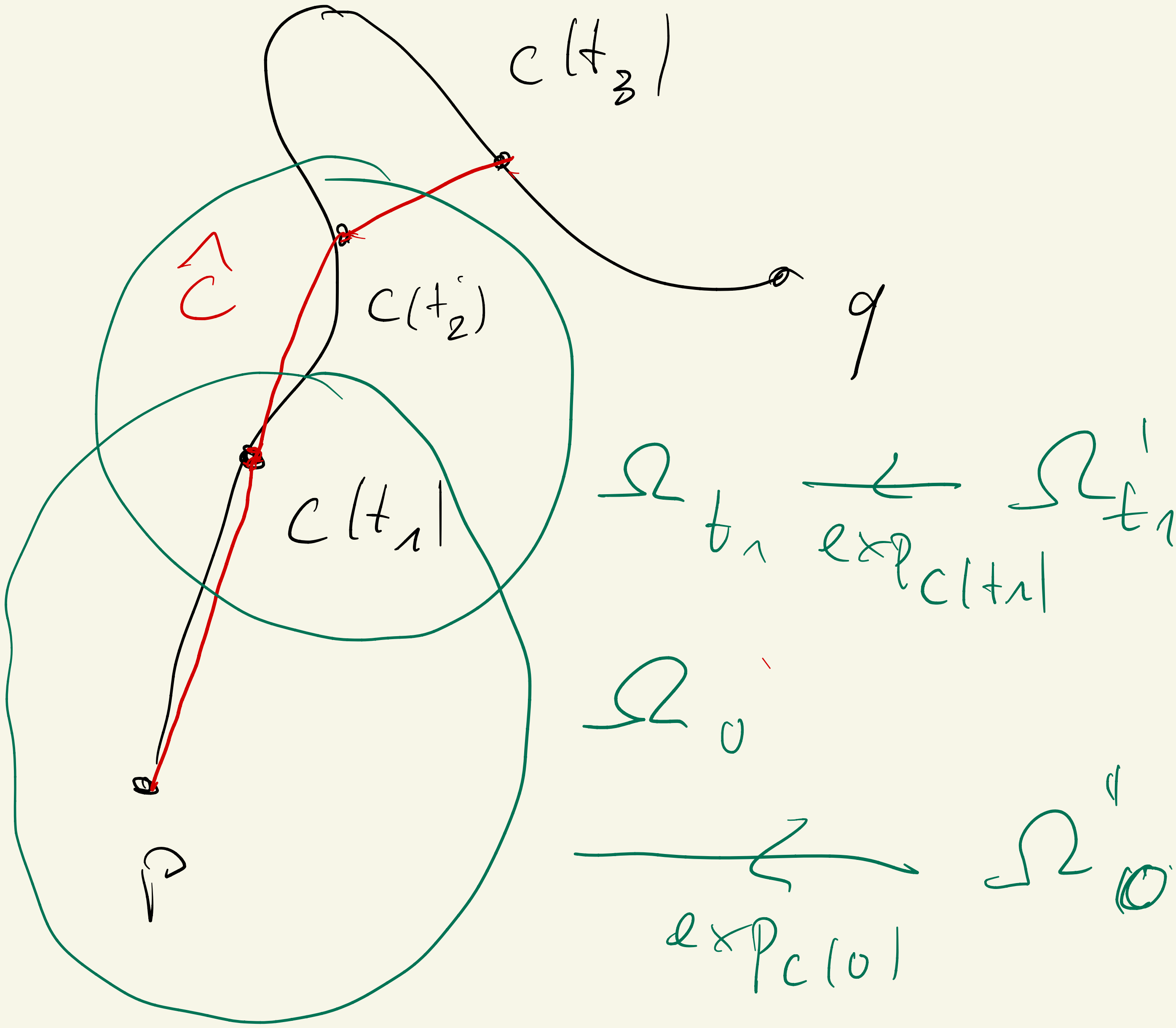
Pf: 1/ a) By assumpt. there is a timelike piecewise C^1-curve $c : [0, b] \longrightarrow M$ from p to q.

Cover image c by starshaped open neighborhoods Ω_t of $c(t)$, s.th. $\Omega_t = \exp_{c(t)}(\Omega_t')$ satisfying assumpt. in Cor 2.13.

As image c is compact, that there $0 = t_0 < t_1 < \ldots < t_k = b$ such that

$$\mathrm{image}\left(c|_{[t_i, t_{i+1}]}\right) \in \Omega_{t_i}$$

$$= \exp_{c(t_i)}\left(\Omega'_{t_i}\right)$$



Define $\hat{c} : [0, b] \to M$

as the curve such that

$$(\exp_{c(t_i)})^{-1} \Big(\underbrace{\hat{c}\big|_{[t_i, t_{i+1}]}}_{\in \Omega_{t_i}} \Big)$$

is the straight line from

$$(\exp_{c(t_i)})^{-1} (c(t_i)) = 0 \in T_{c(t_i)} M$$

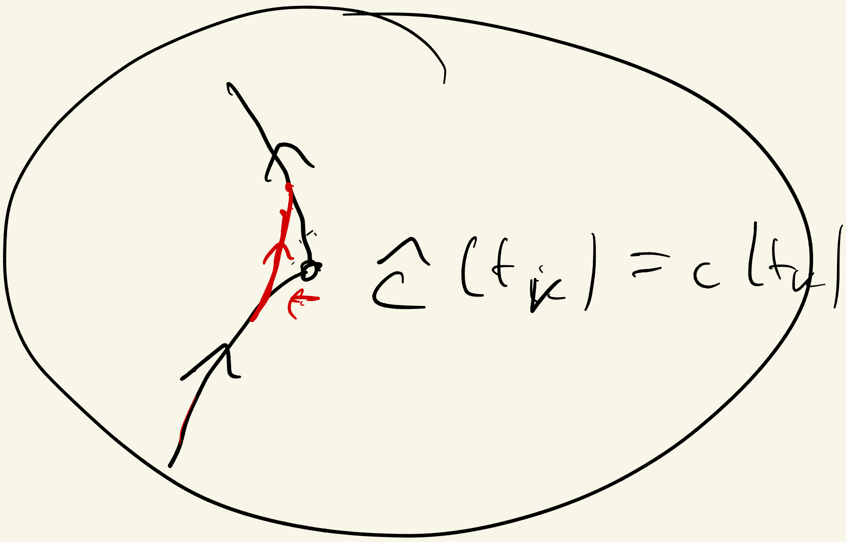
to

$$(\exp_{c(t_i)})^{-1} (c(t_{i+1})) \in T_{c(t_i)} M$$

geodesic piece

$\Rightarrow \hat{c}$ piecewise smooth.

b) (Sketch) smooth out corners at t_k.



$\square$

Prop 7.18

Any compact Lorentzian manifold M carries a smooth periodic timelike curve $c: \mathbb{R} \to M$, i.e. $L > 0$ with $c(t+L) = c(t)$ $\forall t \in \mathbb{R}$ and $\dot{c}(t)$ timelike $\forall t \in \mathbb{R}$.

N