<u>Now</u> (M,g) a Lorentzian manifold (time-oriented)

All curves are (at least) piecewise C^1.

<u>Lemma 2.11</u> (numbers Bär, Lorentz geometry, English)

$p \in M$, $\gamma : [0,b] \to D_p$ such that $\gamma(0) = 0$ $\subset T_pM$

$c := \exp_p \circ \gamma$. Assume c is timelike and future directed. Then $\gamma(t) \in I_+^{T_pM}(0)$ for all $t \in (0,b]$.
past $\underline{-}$

$$\underbrace{I_{\underline{+}}^{T_pM}}_{I_{\underline{+}}}(0) := \{X \in T_pM \mid X \text{ time-like \& future oriented}\}$$
past

$\mathbb{R}^{m,1}$
$\wr\wr$

<u>Proof:</u> (Future case)

a) g_p is a Minkowski metric on T_pM

$$q(x) := g_p(x,x) = -(x^0)^2 + \sum_{i=1}^{m} (x^i)^2, \quad x \in T_pM$$

Assume $\frac{\partial}{\partial x^0}$ future directed $\qquad q : T_pM \to \mathbb{R}$, $x \in T_pM$

$$\underbrace{\operatorname{grad}^{(T_pM, g_p)}}_{= \operatorname{grad}} q|_x = ? \in T_x T_pM$$

For $\forall \xi \in T_x T_pM$

$$g_p(\operatorname{grad} q|_x, \xi) = d_x q(\xi)$$

$$= -2x^0\xi^0 + 2x^1\xi^1 + \dots + 2x^n\xi^n$$

$$= g_p(2x, \xi)$$

$$\operatorname{grad} q|_x = 2x$$

If x time-like, future dir., so is $\operatorname{grad} q|_x$.

Apply Gauß lemma for $V = W = 2x$

$$g_{\exp_p(x)}(d_x \exp_p(V), d_x \exp_p(W)) = g_p(\operatorname{grad} q|_x, \operatorname{grad} q|_x)$$

$$= 4q(x)$$

$$P(x) := \underbrace{d_x \exp_p(V)}_{\in T_{\exp_p x}M} = d_x \exp_p(\underbrace{\operatorname{grad} q}_{\in T_x T_p M})$$ is timelike if $x \in I_+$

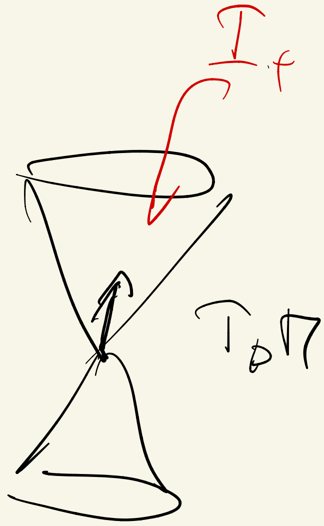
b) Case γ is C^1:

$c = \exp_p \circ \gamma$

$$q(\gamma(0)) = q(0) = 0$$

$$\dot c(0) = \underbrace{(d_0 \exp_p)}_{\hat{=}\, id}(\dot\gamma(0))$$

timelike future



$\exists\, \varepsilon > 0$ s.th.

$\forall t \in (0, \varepsilon]$.

$\gamma(t) \in I_+ = I_+^{T_pM}(0)$ $\qquad q(\gamma(t)) < 0$

To show $\forall t \in (0, b]$: $q(\gamma(t)) < 0$.

Suppose the opposite $\qquad t_1 := \min \underbrace{\{t \in [\varepsilon, b] \mid q(\gamma(t)) = 0\}}_{\neq \emptyset \text{ closed}}$

By the mean value theorem, there is $t_0 \in [0, \varepsilon_1]$ where the minimum of $q \circ \gamma|_{[0, t_1]}$ is attained.

$$\Rightarrow \gamma(t_0) \in I_+ \qquad \frac{d(q\circ\gamma)}{dt}(t_0) = 0$$

$$\frac{d(q\circ\gamma)}{dt} = g_p\left(\operatorname{grad}^{\cancel{\tilde{D}_p}} q_{(\gamma(t))}, \dot\gamma(t)\right)$$

$$\underset{\text{Gauß-Lemma}}{=} g_{c(t)}\Big(d_{\gamma(t)}\exp_p\left(\operatorname{grad} q_{(\gamma(t))}\right),$$

$$\underbrace{d_{\gamma(t)}\exp_p\left(\dot\gamma(t)\right)}_{=\dot c(t)}\Big)$$

$$= g_{c(t)}\Big(\underbrace{P(\gamma(t))}_{\substack{\text{timelike}\\ \text{f. dir.}\\ \text{if } \gamma(t)\in I_+}}, \underbrace{\dot c(t)}_{\substack{\text{ass:}\\ \text{time future}\\ \text{dir.}}}\Big) < 0 \ ↯$$

c) <u>Case</u> γ is piecewise C^1

Let $0 := b_0 < b_1 \ldots < b_N = b$

$\gamma|_{[b_i, b_{i+1}]}$ C^1

$\overset{b)}{\Longrightarrow}$ $\gamma(t) \in \mathcal{I}_f \quad \forall t \in (0, b_1]$.

i) Assume $\gamma(t) \in \mathcal{I}_f^{T_p M}(0) = \mathcal{I}_f$

$\forall t \in (0, b_k]$ for some k.

$$\frac{d}{dt}\Big|_{t=b_k+0} (q \circ \gamma) = g_p(\operatorname{grad} q_{|\gamma(b_k)},$$

$$\dot{\gamma}(b_k+0))$$

Gauß-Lemma

$$\underset{+a)}{=} g_{c(b_k)}(d_{\gamma(b_k)} \exp_p(\operatorname{grad} q_{|\gamma(b_k)}), \ldots$$

$\ldots, d_{\gamma(b_k)} \exp_p(\dot{\gamma}(b_k + 0)))$

$$= g_{c(b_k)}\Big(\underbrace{P(b_k)}_{\text{time-like fut. dir.}}, \underbrace{\dot{c}(b_k+0)}\Big) < 0$$

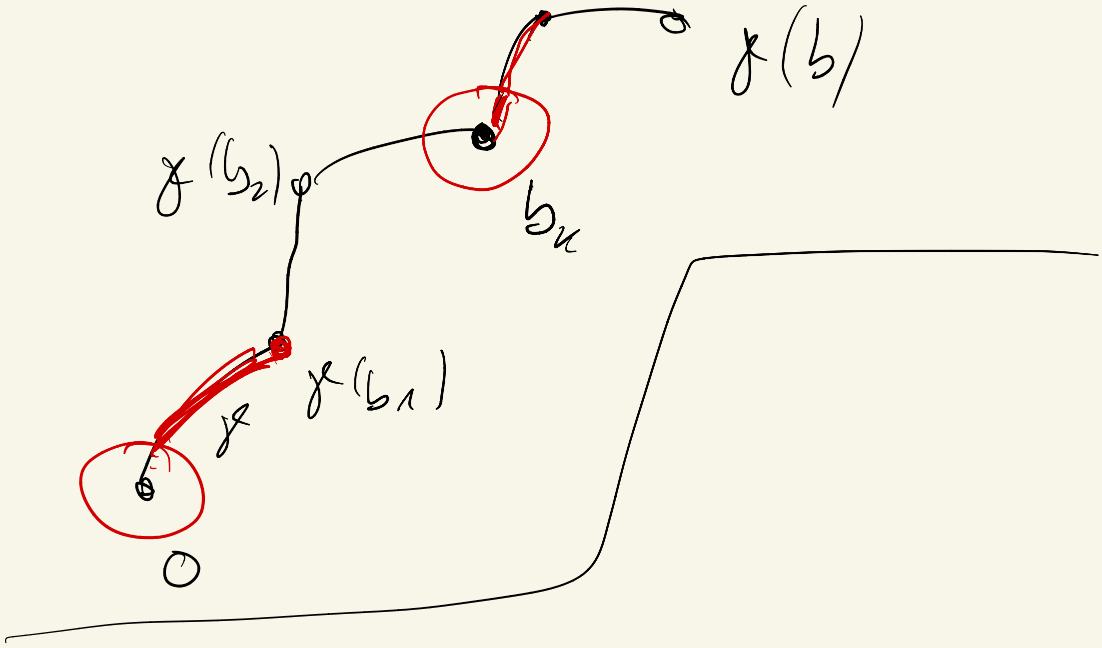
$\Rightarrow \exists\, \varepsilon_0 > 0$ with

$q(\gamma(t))$ for all $t \in [b_k, b_k + \varepsilon_0]$.

$\gamma(t) \in I_+$ " $t \in \ldots$

ii) as in b) we then show

$\gamma(t) \in I_+$ for all $t \in [0, b_{k+1}]$.



Notation 2.12 For $\Omega \Subset M$, $A \subset \Omega$

$$J\,\cancel{I}_+^{\Omega}(A) := \left\{ q \in \Omega \;\middle|\; \exists\, p \in A : \underbrace{p \ll q}_{\text{in } \Omega} \right\}$$

$\gg$

$\leq$

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Cor 2.13 $p \in M$, Ω' starshaped (wrt 0) open nbhd of $0 \in T_pM$

$$\exp_p : \Omega' \longrightarrow \Omega \subset M \quad \text{a diffeo}$$

Then $\overset{J}{\cancel{I}}{}_t^{\Omega}(p) = \exp_p\left(\overset{J}{\cancel{I}}{}_t^{T_pM}(0) \cap \Omega'\right)$