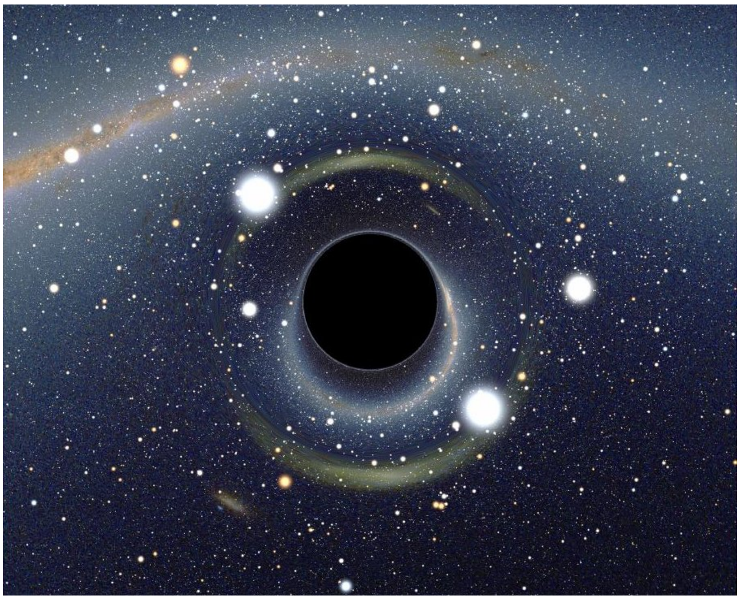
Differential Geometry II
Lorentzian Geometry
Lecture Notes



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2.6 Exponential map and normal coordinates

Let (M, g) be a semi-Riemannian manifold. For some $p \in M$ and $X \in \mathrm{T}_p M$ let $\gamma_X : (a_X, b_X) \to M$ be the geodesic with $\dot\gamma_X(0) = X$ (and thus $\gamma_X(0) = p$), defined on its maximal domain. Obviously we have for $s > 0$

$$\gamma_{sX}(t) = \gamma_X(st), \quad a_{sX} = s^{-1} a_X, \quad b_{sX} = s^{-1} b_X,$$

We set $\mathcal{D}_p := \left\{ X \in \mathrm{T}_p M \mid b_X > 1 \right\}$. In other wors d $X \in \mathrm{T}_p M$ is in $\mathcal{D}_p$ if, and only if, γ_X exists on $[0, 1]$. It follows from above, that $\mathcal{D}_p$ is starshaped with respect to $0 \in \mathrm{T}_p M$, and the dependence on the initial data in the theorem of Picard-Lindelöf shows that $\mathcal{D}_p$ is open in $\mathrm{T}_p M$ and that $\mathcal{D} := \bigcup_{p \in M} \mathcal{D}_p$ is open in $\mathrm{T}M$.

We may define the **exponential map**

$\exp_p := \exp|_{\mathcal{D}_p}$

$$\exp : \mathcal{D} \to M, \qquad X \mapsto \gamma_X(1).$$

$\gamma_{tX}(1)$ $\parallel$

The definition implies $\exp(0 \in \mathrm{T}_p M) = p$ and $\gamma_X(t) = \exp(tX)$ whenever defined, and exp is smooth. We write $\exp_p$ for $\exp|_{\mathcal{D}_p}$. Then with the usual identification $\mathrm{T}_0(\mathrm{T}_p M) \cong \mathrm{T}_p M$. We get $\mathrm{d}_0 \exp_p = \mathrm{id}_{\mathrm{T}_p M}$.

Thus for every $p \in M$ we have an open neighborhood U of 0 in $\mathrm{T}_p M$ such that $\exp_p$ maps U diffeomorphically to $\exp_p(U) \Subset M$. Let $\varphi : \mathbb{R}^{m,k} \to (\mathrm{T}_p M, g_p)$ a linear isometry. Then

$$(\exp_p \circ \varphi|_{\varphi^{-1}(U)})^{-1} : \exp_p(U) \to \varphi^{-1}(U)$$

defines a chart of M around p. Its components are called **normal coordinates** centered in p.

$g_{ij} = \delta_{ij} \varepsilon_i + O(r^2)$

Lemma 2.6.1 (Gauß Lemma)**.** *Let M be a semi-Riemannian manifold, $p \in M$, $X \in \mathcal{D}_p$. Then for any $V = t_0 X, W \in \mathrm{T}_0(\mathrm{T}_pM) \cong \mathrm{T}_pM$ with $t_0 \in \mathbb{R}$, we have*

$T_X(T_pM) = T_0(T_pM) \cong T_pM$

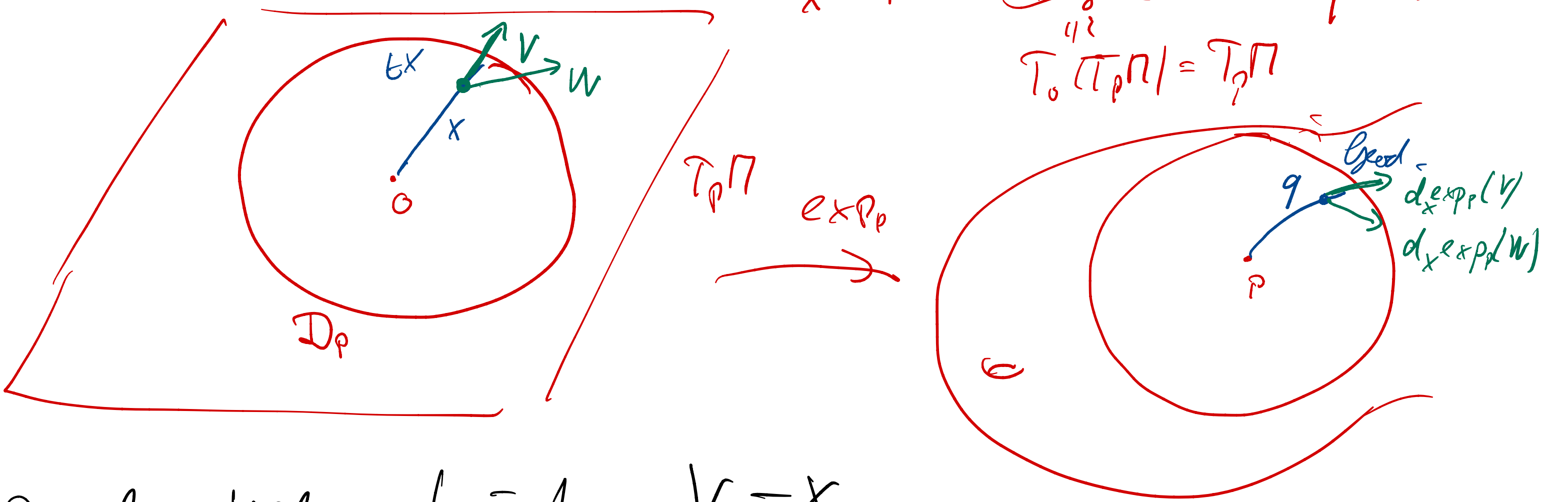
$g_p : T_0(T_pM) \times T_0(T_pM) \to \mathbb{R}$

$$g_q\big(\mathrm{d}_X \exp_p(V), \mathrm{d}_X \exp_p(W)\big) = g_p(V, W).$$

$q = \exp_p(X)$

$d_X \exp_p : \underbrace{T_X(T_pM)}_{\cong\, T_0(T_pM) = T_pM} \to T_q(M)$

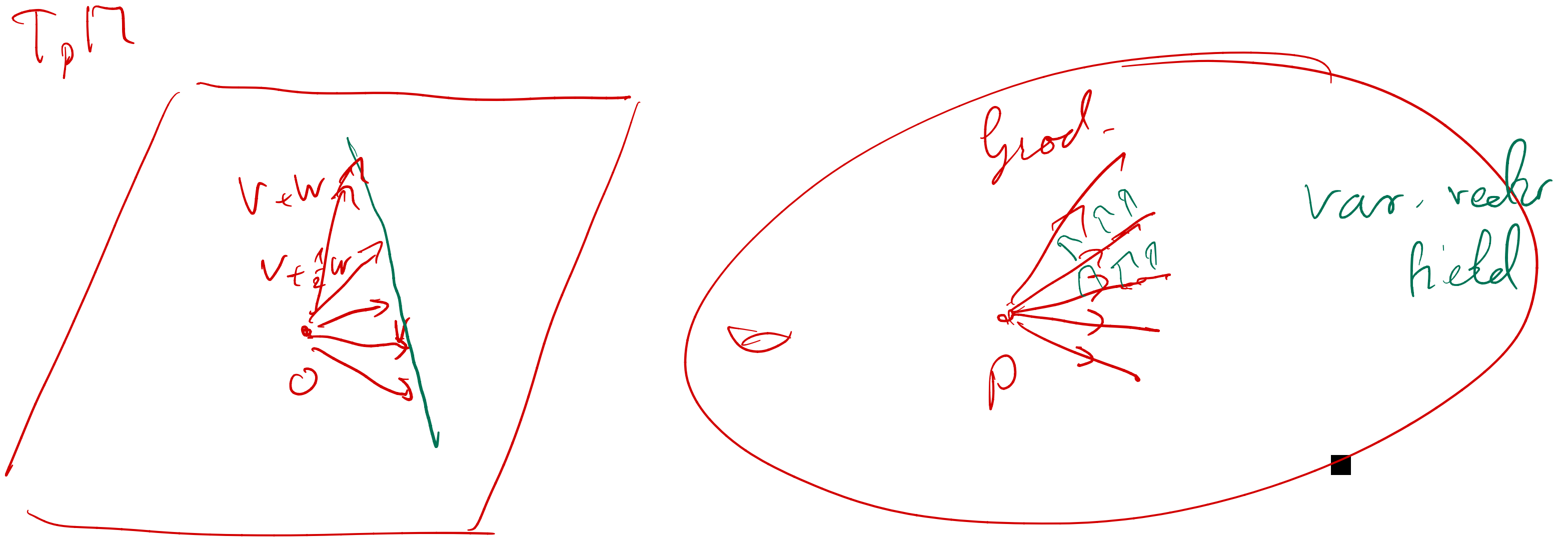
~~**Proof:**~~



Proof: Wlcg $t_0 = 1$, $V = X$

$\varphi : [0,1] \times (-\varepsilon, \varepsilon) \to M$

$(t,s) \longmapsto \exp_p(t(X + sW))$



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$$\frac{\partial \psi}{\partial t}(t,s) = d_{t(X+sW)} \exp_p (X+sW)$$

$$\shortparallel$$

$$\dot{\gamma}_{X+sW}(t) \quad \overset{t=1}{\underset{s=0}{\Longrightarrow}} \quad d_X \exp_p(X) = \frac{\partial \psi}{\partial t}(1,0)$$

$$\overset{t=0}{\underset{\text{sarh.}}{\Longrightarrow}} \quad g\left(\frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial t}\right)\Big|_{t=0} = g(X+sW, X+sW)$$

$$\frac{\partial \psi}{\partial s}(t,s) = d_{t(X+sW)} \exp_p (tW) \overset{t=0}{=} 0$$

$$\Rightarrow \frac{\partial \psi}{\partial s}(1,0) = d_X \exp_p (W)$$

$$0 = \frac{\nabla}{dt} \dot{\gamma}_{X+sW}(t) = \frac{\nabla}{dt} \frac{\partial \psi}{\partial t}$$

$g\left(\frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial t}\right)$ indep. on t!

$$\frac{\partial}{\partial t} \underbrace{g\left(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}\right)\Big|_{s=0}}_{f(t)} = g\left(\underbrace{\frac{\nabla}{dt}\frac{\partial \gamma}{\partial t}}_{=0}, \frac{\partial \gamma}{\partial s}\right)\Big|_{s=0}$$

$$+ g\left(\frac{\partial \gamma}{\partial t}, \frac{\nabla}{dt}\frac{\partial \gamma}{\partial s}\right)\Big|_{s=0}$$

Ex sheet 4 no 3. b)

$$g\left(\frac{\partial \gamma}{\partial t}, \frac{\nabla}{ds}\Big|_{s=0}\frac{\partial \gamma}{\partial t}\right) =$$

$$\frac{1}{2}\frac{\partial}{\partial s}\Big|_{s=0} \underbrace{g\left(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t}\right)}_{\text{cst in } t\text{, say } t=0}$$

$$= g(X+sW, X+sW)$$

$$= g(X,X) + 2s\,g(X,W) + s^2 g(W,W)$$

$$= g(X,W)$$

$f(0) = 0$, $f'(t) = g(X, W)$

$\Rightarrow f(1) = g(X, W) = g(V, W)$

$\parallel$

l.h.s.

$\square$

$$\frac{D}{dt} X := D_{\frac{\partial}{\partial t}} X$$