# Lecture No. 10 11.5.2021

**Prop 2.2** Let $M$ be a Lorentzian mfd. TFAE

(i) $M$ is time-orientable, i.e. there is a map $\zeta : M \to \bigcup_{p \in T} P(T_pM)$, s.th. $\forall p \in M$:

* $\zeta(p)$ is a connected component of
$$\{X \in T_pM \mid g_x(X,X) < 0\}$$
* $\exists$ chart $(x, U, V)$ where $\overset{U}{\mathbb{R}^{n,1}}$ $p \in U \subset M$, $V \subset \mathbb{R}^{n+1}$

$x = (x^0, x^1, \dots, x^n)$

such $\left.\frac{\partial}{\partial x^0}\right|_q \in \zeta(q)$. $\forall q \in U$

(ii) ...

(iii) $M$ admits a smooth timelike vector field

Proved: (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i)

Proof: (i) $\Rightarrow$ (ii)

| dim $M = n = m+1$

~~Consider~~ for every $p \in M$ we consider a chart $p \in U_p \xrightarrow{x_p} V_p$, $U_p \subset M$, $V_p \subset \mathbb{R}^{m+1}$. $M = \bigcup_{p \in M} U_p$

"Überdeckung"

Let $(\eta_p)_{p\in M}$ be a partition of unity associated to covering $(U_p)_{p\in M}$ of $M$,

i.e. 1) $\eta_p : M \to [0,1]$ smooth

2) $\operatorname{supp} \eta_p \subset U_p$, $\operatorname{supp} \eta_p := \overline{\{q \in M \mid \eta_p(q) \neq 0\}}$

3) $(\eta_p)_{p\in M}$ is locally finite

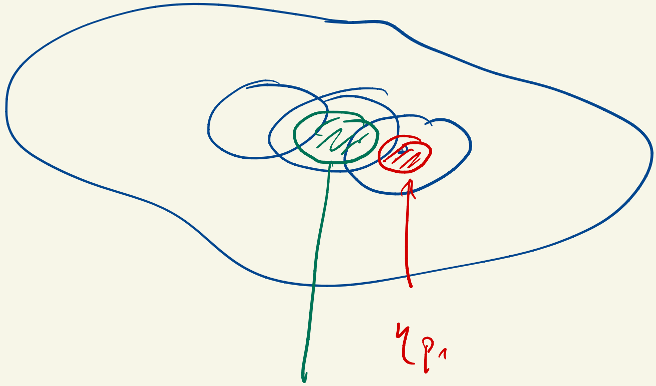
4) $\sum_{p\in M} \eta_p = 1$

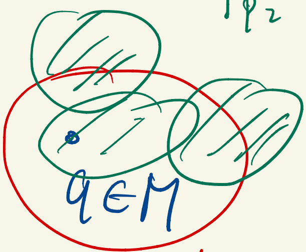
support is in $U_p$, extend it by zero on $M \setminus U_p$

$$X|_q := \sum_{p\in M}^{N} \overbrace{\eta_p(q) \underbrace{\frac{\partial}{\partial x_p^0}\Big|_q}_{\text{well-defined} \in T_qM \text{ if } q \in U_p}}^{\text{support is in } U_p}$$

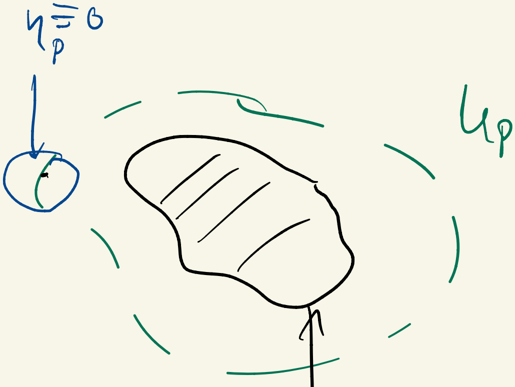
$X \in \Gamma(TM)$

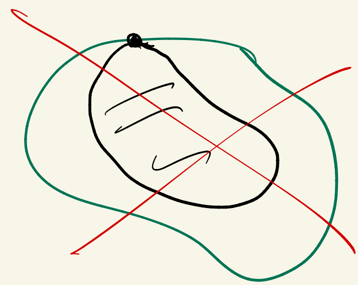
Claim: $X$ is time like




$\forall q \in M \ \exists W_q$ s.th. $\{ p \in M \mid \operatorname{supp} \eta_p \cap W_q \neq \emptyset \}$



$\operatorname{supp} \eta_p = \overline{\{ q \in M \mid \eta_p(q) \neq 0 \}}$

$$\frac{\partial}{\partial x_p^0}\Big|_q \in \mathcal{S}(q) \quad \text{by (i)}$$

Lemma $\Rightarrow$ $X\big|_q := \sum_{p\in\Pi} \eta_p(q) \frac{\partial}{\partial x_p^0}\Big|_q \in \mathcal{S}(q)$ $\square$

as $\mathcal{S}(q)$ is convex

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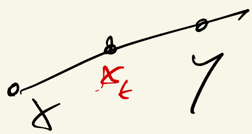
A subset $A \subset V$ is convex if $\forall\, x,y \in A$

vector space

$$\forall\, t\in[0,1]: \underbrace{tx+(1-t)y}_{x_t} \in A$$




If $A$ is convex $t_1,\dots,t_n \in [0,1]$ $\sum t_i = 1$

$x_1,\dots,x_n \in A \Rightarrow \sum_{i=1}^{n} t_i x_i \in A$

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Lemma The connected components of $C = \{ x \in \mathbb{R}^{m,1} \mid \langle\langle x,x\rangle\rangle < 0 \}$ are convex

Proof Let $X, Y \in T$ in the same connect. component

$X = (x^0, \vec{x})$ $\quad |x^0| > \|\vec{x}\|$ $\quad \langle , \rangle$

$Y = (y^0, \vec{y})$ $\quad |y^0| > |\vec{y}|$

w.l.o.g. $x^0 \geq 0$ $\quad y^0 \geq 0$

$x^0 > |\vec{x}|, \; y^0 > |\vec{y}|$ $\quad \alpha, \beta \in \mathbb{R}_{>0}$

CSI

$\langle\langle X, Y \rangle\rangle = -x^0 y^0 + \langle \vec{x}, \vec{y} \rangle \leq -x^0 y^0 + |\vec{x}||\vec{y}| < 0.$

$\langle\langle \alpha X + \beta Y, \alpha X + \beta Y \rangle\rangle = \alpha^2 \langle\langle X, X \rangle\rangle + 2\beta\alpha \langle\langle X, Y \rangle\rangle + \beta^2 \langle\langle Y, Y \rangle\rangle < 0.$

$(\alpha X + \beta Y)_0 = \alpha x^0 + \beta y^0 > 0.$

$\square$

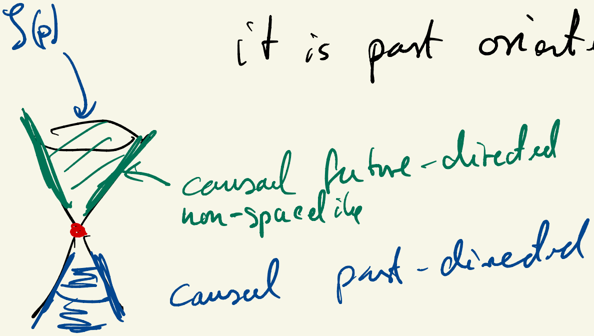
From now on $(M, g, \mathcal{T})$ is a time-oriented Lorentzian manifold.

$\mathcal{T}(p)$ future cone

If $X \in T_pM$ is ~~time~~ causal like

it is future oriented if $X \in \overline{\mathcal{T}(p)}$

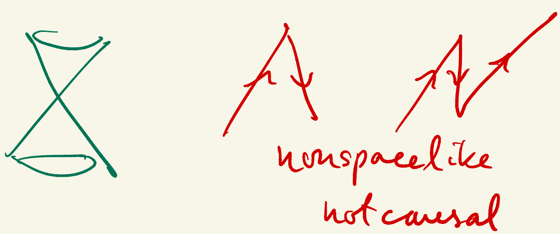
it is past oriented if $X \notin \overline{\mathcal{T}(p)}$

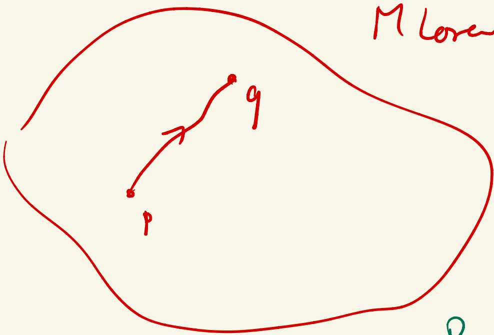



A piecewise $C^1$-curve $c: I \to M$ is <u>causal</u> <u>future oriented</u> (past) if $\forall t \in I$ $\dot{c}(t)$ is causal and (past) future oriented

We say $c$ is causal if $c$ is causal past oriented or causal past-oriented.

$c$ is causal $\overset{\not\Leftarrow}{\Rightarrow}$ $c$ is nonspacelike






$M$ Lorentzian

Physical interpretation

One can send information along future oriented causal curves

Particles or observers can (nonzero rest mass) travel along future-oriented time-like curves.

Notation [No curves $c : \cancel{[a,a] \to M}$ $\cancel{\emptyset \to M}$ allowed. All curves are piecewise $C^1$.

$p \ll q :\Leftrightarrow \exists$ a future-oriented timelike curve from $p$ to $q$.

$p < q :\Leftrightarrow \exists$ a future-oriented causal curve from $p$ to $q$.

$p \leq q :\Leftrightarrow p < q$ or $p = q$

For $A \subset M$ $\quad I_\pm(A) := \{ q \in M \mid \exists p \in A, p \overset{\gg}{\ll} q \}$

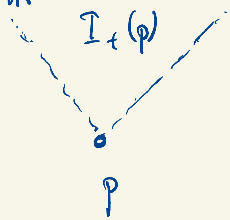
chronological future of $A$
past

$J_\pm(A) := \{ q \in M \mid \exists p \in A, p \overset{\geq}{\leq} q \}$

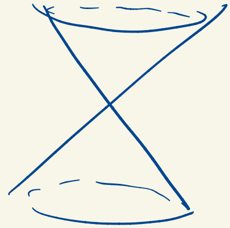
causal future of $A$
past

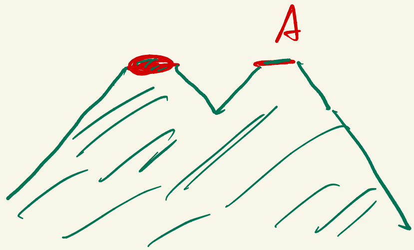
$I_\pm(p) := I_\pm(\{p\})$ $\quad \overset{J}{I}_\pm(A) = \bigcup_{p \in A} \overset{J}{I}_\pm(A)$

$M=\mathbb{R}^{m,1}$




$J_+(p) = \overline{I_+(p)}$






$J_-(A)$

$p \not< p$

$p \not\ll p$

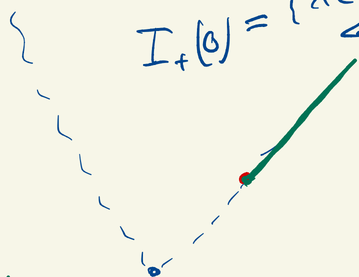
Always $J_+(p) \subset \overline{I_+(p)}$

$I_+(p)$

$M = \mathbb{R}^{m,1} \setminus \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$I_+(0)$

$I_+(0) = \{ x \in \mathbb{R}^{m,1} /$ $\langle\langle x,x \rangle\rangle < 0$ fut. or.$\}$



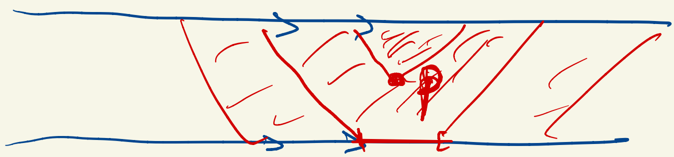
$J_+(0) = \overline{I_+(0)} \setminus \left\{ \begin{pmatrix} t \\ t \\ 0 \\ 0 \end{pmatrix} \middle| \, t \geq 1 \right\}$

$\subsetneq \overline{I_+(0)}^M$

$M = \mathbb{R}^{1,1} / \mathbb{Z} e_0$

$I_+(p) = M = J_+(p)$

$= J(p)$




$p < p$

$p \ll p$

[ $\boxed{p \ll q \Rightarrow p < q}$ ]

$\ll, <, \leq$ transitive

<u>Proposition</u> $p, q, r \in M$

1) $p \ll q$ and $q \leq r \Rightarrow p \ll r$

2) $p \leq q$ and $q \ll r \Rightarrow p \ll r$

<u>Proof</u> Assume 2) holds $\forall M$. Then 1) follows.

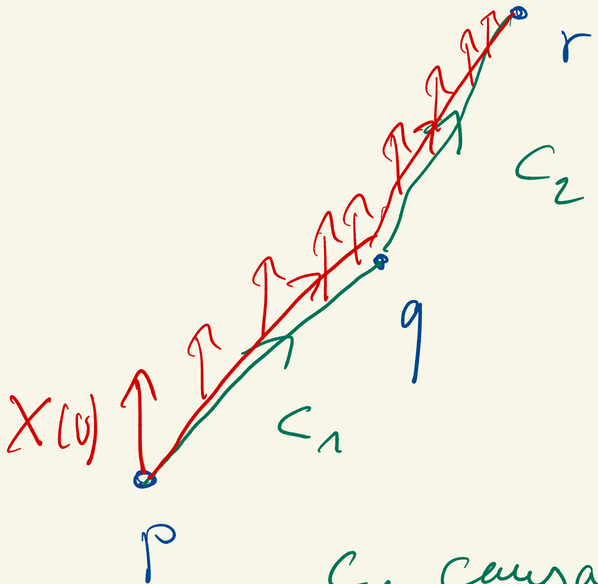
Define $\mathcal{S}^{opp}(p) := \{-X \mid X \in \mathcal{S}(p)\}$

If $p \ll q$ and $q \leq r$, then

$p \gg^{opp} q$ and $q \geq^{opp} r \overset{2) M^{opp}}{\Longrightarrow} p \gg^{opp} r$

$\Rightarrow p \ll r$.

**Proof 2.:** $p \leq q$ and $q \ll r$ wlog $p < q$




$c_2$ timelike futureor.

wlog $c_2 : [1,2] \to M$

$c_1$ causal future oriented

wlog $c_1 : [0,1] \to M$

$$(c_1 * c_2)(t) = \begin{cases} c_1(t) & t \in [0,1] \\ c_2(t) & t \in [1,2] \end{cases}$$

Choose $X(0) \in T_{c_1(0)} M$ timelike, future-oriented

Extend to $X \in \Gamma((c_1 * c_2)^* TM)$ with $\nabla_{\partial_t} X = 0$.

Determine a variation $c(t,s)$ with

$c(t,0) = (c_1 * c_2)(t)$

and $\frac{\partial}{\partial s}\Big|_{s=0} c(t,s) = \begin{cases} t \cdot X|_{c_1(t)} & 0 \leq t \leq 1 \\ (2-t) \cdot X|_{c_2(t)} & 1 \leq t \leq 2 \end{cases}$

$$\boxed{\dot{c}(t,s) = \frac{\partial}{\partial t} c(t,s)}$$

$\mu_s := \max_{t \in [1,2]} g(\dot{c}(t,s), \dot{c}(t,s))_{s=0} < 0$

continuous

Thus $\exists s_0 > 0$ : $\mu_s < 0$ for $0 \leq s \leq s_0$.

For $t \in [0,1]$

$$\frac{\partial}{\partial s} g(\dot{c}(t,s), \dot{c}(t,s))\Big|_{s=0} = 2\, g\left(\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} c(t,s), \frac{\partial}{\partial t} c(t,s)\right)\Big|_{s=0}$$

Exercise sheet 4
Ex no. 3 b)

$$= 2\, g\left(\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}\Big|_{s=0} c(t,s), \frac{\partial}{\partial t} c(t,s)\right)$$

$$= 2\, g\Big(\underbrace{\nabla_{\frac{\partial}{\partial t}} \left(t X|_{c_1(t)}\right)}_{= X|_{c_1(t)} + t \nabla_{\frac{\partial}{\partial t}} X}, \frac{\partial}{\partial t} c(t,s)\Big)$$

$$\nabla_{\frac{\partial}{\partial t}} X = 0 \quad \text{Lemma}$$

$$= 2\, g\Big(\underbrace{X|_{c_1(t)}}_{\text{time-like future}}, \underbrace{\dot c_1(t)}_{\text{causal future oriented}}\Big) < 0$$

$$\frac{\partial}{\partial s} c(t,s)\Big|_{s=0} = t X|_{c_1(t)}$$

As $g(\dot c(t,0), \dot c(t,0)) \le 0$ ~~and~~ there is $s_1 \in (0, s_0)$ such that $g(\dot c(t,s), \dot c(t,s)) < 0$ $\forall t \in [1,2]$ $\forall s \in (0, s_1)$ $\square$

$=0$

Why $X$ timelike

$$\frac{\partial}{\partial t} g(X, X) = 2 g\Big(\underbrace{\nabla_{\frac{\partial}{\partial t}} X}_{=0}, X\Big) = 0$$

by continuity future-oriented.

Lemma: Let $X, Y \in \mathbb{R}^{n,1}$ be causal with the same time-orientation. Then $g(X,Y) \leq 0$. Furthermore if $g(X,Y) = 0$, then $X, Y$ are lightlike and $X, Y$ are linearly dependent

Pf: $X = (x^0, \vec{x})$, $Y = (y^0, \vec{y})$ wlog $x^0 > 0$, $y^0 > 0$

~~then~~ $x^0 \geq |\vec{x}|$ $\quad y^0 \geq |\vec{y}|$

$$g(X,Y) = -x^0 y^0 + \langle \vec{x}, \vec{y} \rangle \overset{CSI}{\leq} -x^0 y^0 + |\vec{x}||\vec{y}| \leq 0$$

Equality Then $\vec{x} = \lambda \vec{y}$ $\quad x^0 = |\vec{x}|$

$y^0 = |\vec{y}|$

$\Rightarrow X = \lambda Y$ $\quad \square$

**Cor** For $A \subset M$

$$I_\varepsilon(A) = I_\varepsilon(I_\varepsilon(A))$$

$$= J_\varepsilon(I_\varepsilon(A)) = I_\varepsilon(J_\varepsilon(A))$$

$$\subset J_\varepsilon(J_\varepsilon(A)) \subset J_\varepsilon(A)$$