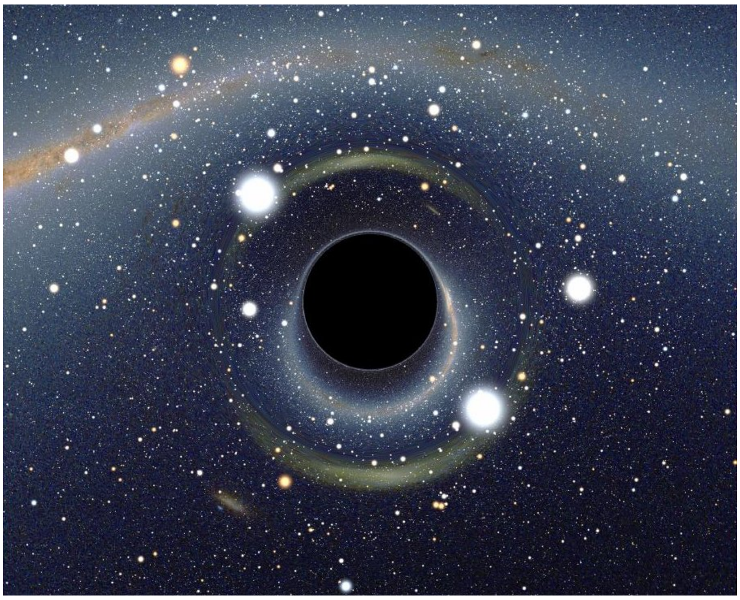
Differential Geometry II
Lorentzian Geometry
Lecture Notes



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Presentation Version

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Recapitulation before the lecture on 10.05.:

$B \times F$

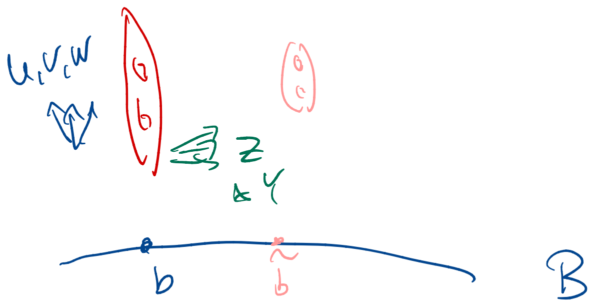
Let $\check{g}$ be a semi-Riemannian metric on B.
Let $\hat{g}^b = w^2 g^F$ be a family of semi-Riemannian metrics on F.
$w \in \mathcal{C}^\infty(B)$, $w > 0$

warped product

$$g_w = \check{g} + w^2 g^F$$

horizontal = tangential to B-direction

vertical = tangential to F-direction



Theorem 2.5.13. *(Riemann curvature tensor for warped products) Let g_w be a warped product on $B \times F$ with the notation from above. Let X, Y, Z be horizontal and let U, V, W be vertical vector fields on M. Then*

$B \times \{f\}$ $\quad$ $\nabla \mathrm{II} = 0$

$$R(X,Y)Z = \check{R}(X,Y)Z \tag{2.5.1}$$

$$\begin{aligned} R(X,U)Y &= \frac{(\nabla^2 w)(X,Y)}{w} U \\ &= \big((\nabla^2 \log w)(X,Y) + (\partial_X \log w)(\partial_Y \log w)\big) \end{aligned} \tag{2.5.2}$$

$$R(U,V)X = 0 \tag{2.5.3}$$

$$R(X,Y)U = 0 \tag{2.5.4}$$

$$R(X,U)V = -\frac{g_w(U,V)}{w} \nabla_X(\operatorname{grad} w) \tag{2.5.5}$$

$$\begin{aligned} R(U,V)W &= \hat{R}(U,V)W \\ &\quad - \frac{g(\operatorname{grad} w, \operatorname{grad} w)}{w^2}\big(\hat{g}(V,W)U - \hat{g}(U,W)V\big). \end{aligned} \tag{2.5.6}$$

$R(U,v)v$

ver hor

Theorem 2.5.13. *(Riemann curvature tensor for warped products) Let g_w be a warped product on $B \times F$ with the notation from above. Let X, Y, Z be horizontal and let U, V, W be vertical vector fields on M. Then*

$X = v = Y$

$-R(U,X)Y =$

$$R(X,Y)Z = \check{R}(X,Y)Z \tag{2.5.1}$$

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$$R(X,U)V = -\frac{g_w(U,V)}{w} \nabla_X(\operatorname{grad} w) \tag{2.5.5}$$

$$\begin{aligned} R(U,V)W &= \hat{R}(U,V)W \\ &\quad - \frac{g(\operatorname{grad} w, \operatorname{grad} w)}{w^2} \big(\hat{g}(V,W)U - \hat{g}(U,W)V\big). \end{aligned} \tag{2.5.6}$$

"(2.5.5)": $R(X,U)V$ is horizontal because of

$$g_w(R(X,U)V,W) = g_w(R(V,W)X,U) \overset{(2.5.3)}{=} 0.$$

For the horizontal part we calculate

$$\begin{aligned} g_w(R(X,U)V,Y) &= -g_w(R(X,U)Y,V) \\ &\overset{(2.5.2)}{=} -g_w\left(\frac{(\nabla^2 w)(X,Y)}{w} U, V\right) \\ &\overset{(*)}{=} -\frac{g_w(U,V)}{w} g_w(\nabla_X(\operatorname{grad} w), Y), \end{aligned}$$

where we used at $(*)$:

$$\begin{aligned} (\nabla^2 f)(X,Y) &= (\nabla_X \mathrm{d}f)(Y) = \big(\nabla_X (\operatorname{grad} f)^\flat\big)(Y) \\ &= \big(\nabla_X \operatorname{grad} f\big)^\flat(Y) = g_w(\nabla_X \operatorname{grad} f, Y) \end{aligned}$$

Theorem 2.5.14 (Ricci curvature tensor for warped products)**.** *Let g_w be a warped product on $B \times F$ with the notation from above. Let X, Y be horizontal and let U, V be vertical vector fields on M. Let $\ell := \dim F$. Then*

$\Sigma g(R(X, e_i)e_i, Y)$
e_i

$$\mathrm{ric}(X,Y) = \check{\mathrm{ric}}(X,Y) - \ell \frac{(\nabla^2 w)(X,Y)}{w} \tag{2.5.7}$$

$$\mathrm{ric}(X,V) = 0 \tag{2.5.8}$$

$$\mathrm{ric}(U,V) = \mathrm{ric}^F(U,V) + g(U,V)h \tag{2.5.9}$$

where

$$h = \frac{\Delta w}{w} - (\ell - 1)\frac{g(\operatorname{grad} w, \operatorname{grad} w)}{w^2} = \frac{\Delta w^\ell}{\ell w^\ell}.$$

$$\mathrm{scal} = \check{\mathrm{scal}} + \frac{\mathrm{scal}^F}{w^2} + h_1, \tag{2.5.10}$$

where

$$h_1 = 2\ell \frac{\Delta w}{w} - \ell(\ell - 1)\frac{g(\operatorname{grad} w, \operatorname{grad} w)}{w^2} = \frac{4\ell}{\ell + 1}\frac{\Delta(w^{(\ell+1)/2})}{w^{(\ell+1)/2}}.$$

Proof:
"(2.5.7)": This follow immediately from contracting (2.5.1) and (2.5.2).

"(2.5.8)": This follow immediately from contracting (2.5.3) and (2.5.4).

"(2.5.9)": The formula with the first expression for h follows immediately from contracting (2.5.5) and (2.5.6). Furthermore we calculate

$$\begin{aligned} (\nabla^2 w^\ell)(X,Y) &= \nabla_X(\mathrm{d}w^\ell)(Y) = \ell \nabla_X(w^{\ell-1}\,\mathrm{d}w)(Y) \\ &= \ell(\ell-1)w^{\ell-2}\,(\mathrm{d}w \otimes \mathrm{d}w)(X,Y) + \ell w^{\ell-1}(\nabla^2 w^\ell)(X,Y) \end{aligned}$$

$$\Delta(w^\ell) = -\varepsilon_j \nabla^2_{e_j,e_j}(w^\ell) \qquad \ell \in \mathbb{R}$$

$$= -\varepsilon_j \nabla_{e_j}(\nabla_{e_j} w) + \varepsilon_j \nabla_{\nabla_{e_j} e_j} w$$

$$\nabla_Y(w^\ell) = \ell\, w^{\ell-1}(\nabla_Y w)$$

$$\nabla_X \nabla_Y(w^\ell) = \ell(\ell-1)\, w^{\ell-2}(\nabla_X w)(\nabla_Y w) + \ell\, w^{\ell-1} \nabla_X(\nabla_Y w)$$

$$\Rightarrow \Delta(w^\ell) = -\ell(\ell-1)\, w^{\ell-2} \underbrace{\varepsilon_i(\nabla_{e_i} w)(\nabla_{e_i} w)}_{g(\operatorname{grad} w,\, \operatorname{grad} w)} + \ell\, w^{\ell-1} \Delta w$$

which yields by contraction

$$\begin{aligned}
\Delta w^\ell &= -\sum_{i=1}^{\dim M} \epsilon_i (\nabla^2 w^\ell)(e_i, e_i) \\
&= -\ell \sum_{i=1}^{\dim M} \epsilon_i \big((\ell-1) w^{\ell-2} (\mathrm{d}w \otimes \mathrm{d}w)(e_i, e_i) + w^{\ell-1} (\nabla^2 w)(e_i, e_i)\big) \\
&= -\ell(\ell-1) w^{\ell-2} g(\operatorname{grad} w, \operatorname{grad} w) + \ell w^{\ell-1} (\Delta w)
\end{aligned}$$

and this yields the other formula for h.
"(2.5.10)": This is obtained from (2.5.7) and (2.5.9) by contraction, and similar calculations with $\Delta(w^\alpha)$ as above. ∎

Example 2.5.9 cont'd. *Let us discuss these formulas for Robertson-Walker spacetimes. Let $\hat{g}$ be a Riemannian metric of constant sectional curvature κ on F, and let again $w : (a, b) \to \mathbb{R}_{>0}$ be smooth. We consider on $M = (a, b) \times F$ the semi-Riemannian metric*

$B = (a, b)$

$\{b\} \times F$

$$g = -\mathrm{d}t \otimes \mathrm{d}t + w(t)^2 \hat{g}.$$

$M = (a, b) \times F$

Let $U, V, W \in \mathscr{X}(M)$ be vertical vector fields and $\nu := \partial_t$, which is a horizontal vector field with $g(\nu, \nu) = -1$. Note that $\operatorname{grad} w = -w' \nu$ and $\check{\Delta} = \frac{\partial^2}{\partial t^2}$. From Theorem 2.5.13 we directly get

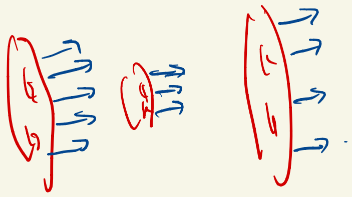
$$R(U, \nu)\nu = -\frac{w''(t)}{w(t)} U \tag{2.5.11}$$

$$R(U, V)\nu = 0 \tag{2.5.12}$$

$$R(\nu, U)V = \frac{g_w(U, V)}{w(t)} w''(t)\, \nu \tag{2.5.13}$$

$$R(U, \nu)\nu = -\frac{w''(t)}{w(t)} U \tag{2.5.14}$$

$$R(U, V)W = \frac{\kappa + w'(t)^2}{w(t)^2} \big(\hat{g}(V, W)U - \hat{g}(U, W)V\big). \tag{2.5.15}$$



ν unit normal field to each $\{s\}\times F$ horizontal

$B=(a,b)$

$g(\nu,\nu)=-1$

and for $\ell = \dim F$

$$\begin{aligned}
\mathrm{ric}(\nu,\nu) &= -\ell\,\frac{w''(t)}{w(t)} && (2.5.16)\\
\mathrm{ric}(\nu,V) &= 0 && (2.5.17)\\
\mathrm{ric}(U,V) &= \left(\frac{w''(t)}{w(t)} + \frac{(\ell-1)\big(\kappa + w'(t)^2\big)}{w^2(t)}\right) g(U,V). && (2.5.18)
\end{aligned}$$

$$\mathrm{scal} = \ell\left(2\,\frac{w''(t)}{w(t)} + \frac{(\ell-1)\big(\kappa + w'(t)^2\big)}{w(t)^2}.\right) \tag{2.5.19}$$

The formulas for rotationally symmetric Riemannian metrics, see Example 2.5.10, coincide with these formulae up to some signs.

3 Lorentzian manifolds

3.1 General Relativity

The basic idea of general relativity is that matter curves the space. For physical justification of the precise laws, we refer to physics textbooks. The goal of this subsection, is to explain the required background for our lecture.

We will need some further definitions. Let (M, g) be an m-dimensional semi-Riemannian manifold. For $h \in \Gamma(\mathrm{T}^{0,s}M)$, $s \geq 2$, one defines the **(metric) contraction** of the first and second component as

Scal $= g R(\cdot,\cdot,\cdot,\cdot)$

$$\sum_{j=1}^{m} \epsilon_j h(e_j, e_j, \bullet, \ldots, \bullet), \in \Gamma(\mathrm{T}^{0,s-2}M)$$

where $(e_1, \ldots, e_m)$ is a generalized orthonormal basis. One can similarly define contractions in other arguments. In the case $h \in \Gamma(\mathrm{T}^{0,2}M)$ we also used the term **metric trace**, written as $\mathrm{tr}^g h$, for the contraction of the two components. The (usual) trace and the metric trace are related by the formula: if $H \in \Gamma(\mathrm{End}(\mathrm{T}M))$, then

$$\mathrm{tr}\, H = \mathrm{tr}^g\big(g(H(\bullet), \bullet)\big).$$

For a symmetric $(0, s)$-tensor h on M, one defines its **divergence** $\mathrm{div}\, h \in \Gamma(\mathrm{T}^{0,s-1}M)$ as the metric trace of ∇h in the first two components, i. e.,

$$\forall p \in M : \forall X_1, \ldots, X_{s-1} \in \mathrm{T}_pM :$$
$$\mathrm{div}\, h(X_1, \ldots, X_{s-1}) := \sum_{j=1}^{\dim M} \epsilon_j (\nabla_{e_j} h)\big(e_j, X_1, \ldots, X_{s-1}\big).$$

 $\mathrm{div}\, \alpha = \mathrm{div}(\alpha^\#)$

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Definition 3.1.1. *For a semi-Riemannian metric* g*, the* **Einstein tensor** *is defined as*

$$G := \mathrm{ric} - \frac{1}{2}\,\mathrm{scal}\cdot g$$

$\mathrm{div}(\mathrm{ric}) = \frac{1}{2} d\,\mathrm{scal} = \frac{1}{2}\mathrm{div}(\mathrm{scal}\cdot g)$

$\mathrm{div}\, G = 0$

A key concept of general relativity is that our universe is modeled by an $m = (n+1)$-dimensional Lorentzian manifold that satisfies the **Einstein equations** which are given by

$\int_M \mathrm{scal}^g\, d\mathrm{vol}^g$

$$G + \Lambda g = 8\pi T, \tag{3.1.1}$$

where T is the energy-momentum tensor, and where Λ is a real constant.

The relevant case for traditional general relativity is $m = 3 + 1$, but in string or super-string theories one also considers other dimensions.

When Einstein published the mathematical framework of general relativity in 1915, he was able to use the concepts of Riemannian geometry established by Riemann, Ricci-Curbastro and many others, at the end of the 19th century, by adding signs. It took a while until Einstein got the right type of equations. In fact Einstein got the right equations after he had an extensive exchange with the famous mathematician David Hilbert. At first, Einstein thought the left hand side should be ric, but this led to the contradiction $\mathrm{div}\, T \neq 0$ – see below. After discussions with Hibert, he assumed G for the left hand side, i. e., as above with $\Lambda = 0$. Later on he realized that $\Lambda \neq 0$ is also possible. Later again, he regretted this a lot, he said that allowing $\Lambda \neq 0$ was “die größte Eselei meines Lebens”. So he came back to G. Nowadays physicists are convinced that $\Lambda \neq 0$, and it is called the **cosmological constant** or the **dark energy**.

The tensor T describes the effect of the matter and of the fields. Obviously, as long as we do not have information about T, the statement of the Einstein equations is void, but the explicit form of the T-tensor is involved. Let us mention that one usually describes the energy-momentum tensor by writing down the "action functional" and then by deriving with respect to perturbations of the semi-Riemannian metric. In fact, it is even too early to write down the Lagrange functional even for simple systems. We might explain this in more details later on, but even if we just want to describe an electron coupled to an electromagnetic field we should discuss spinors, spin^c-structures, the Dirac operator,....

Thus, we restrict to summarize some simple properties of T.

(1) If the spacetime describes a vacuum, then $T = 0$.

(2) For all spacetimes we have $\operatorname{div} T = 0$.

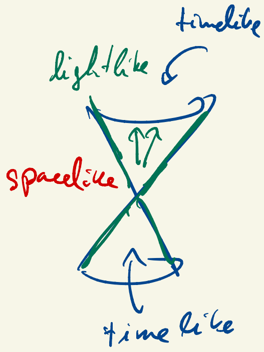
(3) It is consensus that the energy-momentum tensor satisfies several positivity assumptions: $\swarrow T_pM$

- the **dominant energy condition**: If X, Y are timelike (or causal) tangent vectors with the same basepoint and the same time orientation, then $T(X, Y) \geq 0$.
- the **weak energy condition**: $T(X, X) \geq 0$ for all timelike (or causal) $X \in \mathrm{T}M$.
- the **null energy condition**: $T(X, X) \geq 0$ for all lightlike $X \in \mathrm{T}M$.

Property (2) can be seen from the experimental or the mathematical perspective. It can be seen as the conservation of energy and momentum density, which one can verify experimentally. On the other hand, if we know how to derive the Einstein equations from the action func-

$(T_pM, g_p) \underset{\text{isometric}}{\cong} \mathbb{R}^{n,1}$

γ timelike $\Leftrightarrow \langle\langle x, \gamma\rangle\rangle < 0$



causal $\Leftrightarrow$ timelike or lightlike

$\Leftrightarrow \langle\langle x, \gamma\rangle\rangle \leq 0$

tional, it can be proven mathematically. It also provides the contradiction mentioned above: we have seen in Exercise Sheet 3, Exercise 3 (b) that $\operatorname{div}\operatorname{ric} = \frac{1}{2}\mathrm{d}\,\mathrm{scal}$ which is non-zero in general for solution of $\mathrm{ric} = 8\pi T$.

Property (3) is physically interpreted is non-negative energy density in different senses. The dominant energy condition implies the weak energy condition, and the weak energy condition implies the null energy condition.

Remark 3.1.2. Another positivity condition found in the literature is the **strong energy condition**: $(T - \frac{1}{m-2}(\operatorname{tr} T)g)(X,X) \geq 0$ for all timelike (or causal) $X \in \mathrm{T}M$. Other than indicated by its name, it does not imply the weak energy condition. Furthermore there are matter models for which it is not satisfied.

Remark 3.1.3. By taking the metric trace on both sides, (3.1.1) implies

$$\operatorname{tr}^g G + m\Lambda = 8\pi \operatorname{tr}^g T,$$

~~and this can be resolved as an expression~~ for ric

Einstein equation

$\mathrm{ric} - \frac{1}{2}\mathrm{scal}\, g + \Lambda g = 8\pi T$

$$\begin{aligned}\mathrm{ric} &= \frac{1}{2}\mathrm{scal}\cdot g + 8\pi T\\ &= \frac{m}{m-2}\Lambda\cdot g + 8\pi\left(T - \frac{1}{m-2}(\operatorname{tr}^g T)\cdot g\right)\end{aligned}$$

$\Leftrightarrow$ Einstein equation

This is in fact an equivalent formulation for (3.1.1).

In particular, for $\Lambda = 0$ the strong energy condition is equivalent to *non-positive* Ricci curvature in all timelike directions, i. e., $\mathrm{RIC}(X) \leq 0$ for all timelike X.

The Einstein equation as a flow.

Roughly speaking: the Einstein equation describes the evolution of the spacetime, $\operatorname{div} T = 0$ describes the evolution of matter. This rough picture is not complete, as $\operatorname{div} T = 0$ does not describe fully the evolution of matter.

The spacetime of our universe has further structure: a time-orientation and a causal structure which we will describe below.

A particle (or an observer) in this spacetime is given by a curve $c : (a, b) \to M$ with $g(\dot{c}(\tau), \dot{c}(\tau)) \leq 0$, and the case $g(\dot{c}(\tau), \dot{c}(\tau)) = 0$ is only allowed for particles with zero rest mass.

3.2 Causality

In this subsection we will follow [5, Section 2] up to ??.