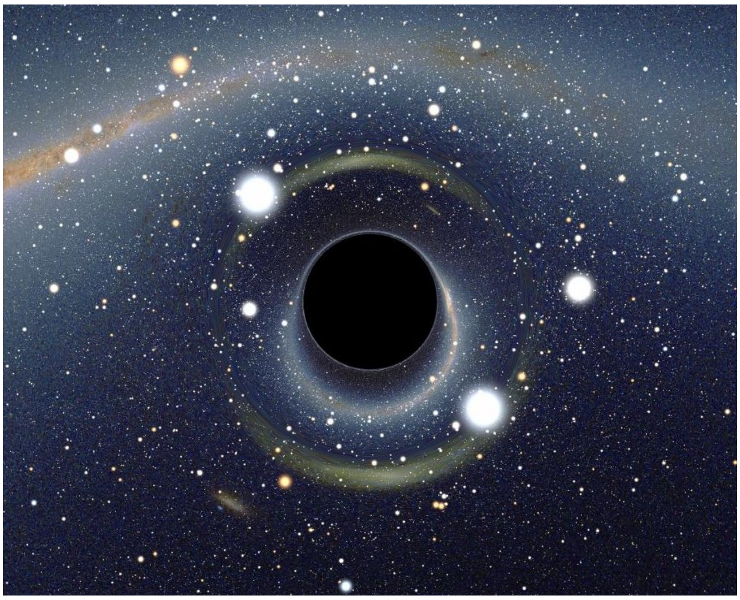
Differential Geometry II
Lorentzian Geometry
Lecture Notes



Prof. Dr. Bernd Ammann

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University of Regensburg

Presentation Version

Version of May 5, 2021

Recapitulation before the lecture on 05.05.:

Generalized warped product metrics.

$M = B \times F$

$\check{g}$ semi-Riemannian metric on B, also $\check{g} := (\pi^B)^* \check{g} \in \Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M)$

For any $b \in B$ have a metric $\hat{g}^b$, semi-Riemannian metric on F, join together to $\hat{g} \in \Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M)$

$$g_{\mathrm{gwp}} = \check{g} + \hat{g}$$

$$g^{\mathrm{gwp}}_{(b,f)}\big((X_1, Y_1), (X, Y_2)\big) := \check{g}_b(X_1, X_2) + \hat{g}^b_f(Y_1, Y_2).$$

Fix f $\quad$ $\underbrace{T_f F^b}$

$b \longmapsto \hat{g}^b_f : T_f F \times T_f F \to \mathbb{R}$

$B \longrightarrow \in \underbrace{\text{Bilin.}(T_f F, T_f F)}_{\text{vect}}$

$B \xrightarrow{\hat{g}} \Gamma(T^*F \otimes T^*F)$

Goal.

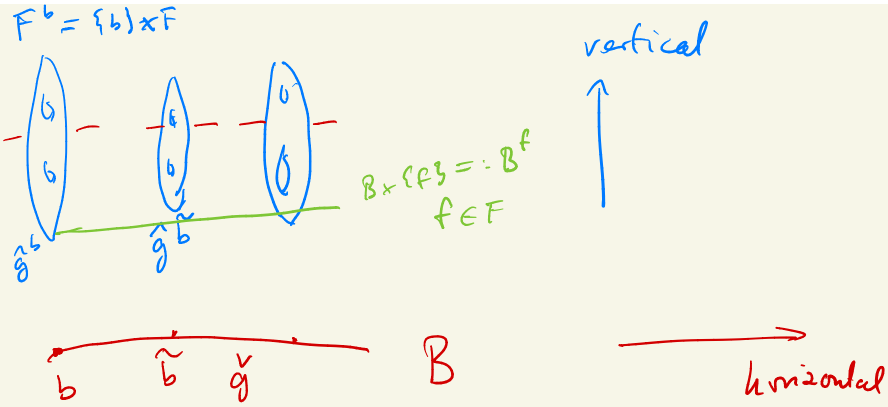
Determine the curvature of (M, g_{gwp})

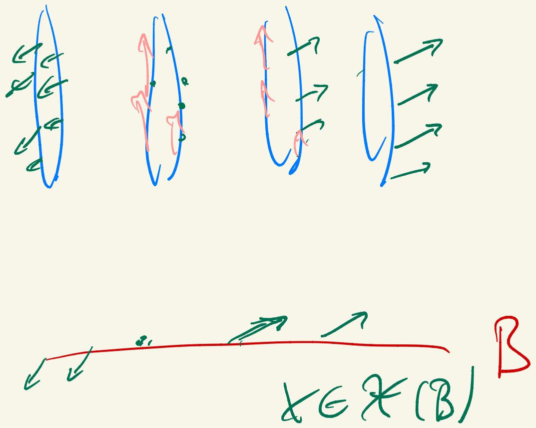
Proposition 2.5.2. *Let g_{gwp} a generalized warped product metric on $M = B \times F$ as above.*

(a) *For each $b \in B$ the second fundamental form of $F^b := \{b\} \times F$ in (M, g_{gwp}) satisfies for all $U, V \in \mathscr{X}(F^b)$ and all $X \in \mathrm{T}_b B \subset \mathrm{T}_{(b,f)} M$*

$$g_{\mathrm{gwp}}\big(\vec{\mathrm{II}}^{F^b}(U, V), X\big) = -\frac{1}{2}(\partial_X \hat{g})(U, V)$$

(b) *For each $f \in F$ the second fundamental form of $B^f := B \times \{f\}$ in (M, g_w) vanishes.*


$F^b = \{b\} \times F$
vertical
$B \times \{f\} =: B^f$
$f \in F$
$\hat{g}^b$
$\hat{g}\,\tilde{b}$
b
$\tilde{b}$
$\check{g}$
B
horizontal


$X \in \mathfrak{X}(B)$
B

$$T^{r,s}M = \underbrace{TM \otimes \dots \otimes TM}_{r\text{-times}} \otimes \underbrace{T^*M \otimes \dots \otimes T^*M}_{s\text{-times}}$$

Remark 2.5.3. Before we prove the proposition, let us explain how to pullback tensors $\Gamma(\mathrm{T}^{r,s}B)$ to tensors in $\Gamma(\mathrm{T}^{r,s}M)$.
For $r = 0$ this is just the standard pullback of $(0,s)$-tensors aling π^B, i. e., for $\alpha \in \Gamma(\mathrm{T}^{0,s}B)$ we defined

$$\left(\pi^B\right)^* \alpha(X_1, \dots, X_s) := \alpha(\mathrm{d}\pi^B(X_1), \dots, \mathrm{d}\pi^B(X_s)).$$

The tangent space of M in $x = (b, f) \in M$ can be identified as follows

$$\mathrm{T}_{(b,f)}(B \times F) = \mathrm{T}_b B \oplus \mathrm{T}_f F.$$

For a vector field $X \in \mathscr{X}(B)$ we define $(\pi^B)^* X \in \mathscr{X}(B \times F)$ by

$$\left(\pi^B\right)^* X\big|_{(b,f)} := \left(X\big|_b, 0\right).$$

By additionally requiring linearity and compatibility with tensoring

$$\left(\pi^B\right)^* (\tau_1 \otimes \tau_2) = \left(\left(\pi^B\right)^* \tau_1\right) \otimes \left(\left(\pi^B\right)^* \tau_2\right)$$

we obtain

$$\left(\pi^B\right)^* : \Gamma(\mathrm{T}^{\bullet,\bullet}B) \to \Gamma(\mathrm{T}^{\bullet,\bullet}M).$$

Similarly we obtain

$$\left(\pi^F\right)^* : \Gamma(\mathrm{T}^{\bullet,\bullet}F) \to \Gamma(\mathrm{T}^{\bullet,\bullet}M).$$

$\check{g}$ "=" $(\pi^B)^* \check{g}$

Proposition 2.5.2. *Let g_{gwp} a generalized warped product metric on $M = B \times F$ as above.*

(a) *For each $b \in B$ the second fundamental form of $F^b := \{b\} \times F$ in (M, g_{gwp}) satisfies for all $U, V \in \mathscr{X}(F^b)$ and all $X \in \mathrm{T}_b B \subset \mathrm{T}_{(b,f)} M$*

$$g_{\mathrm{gwp}}\big(\vec{\mathrm{II}}^{F^b}(U,V), X\big) = -\frac{1}{2}(\partial_X \hat{g})(U,V)$$

(b) *For each $f \in F$ the second fundamental form of $B^f := B \times \{f\}$ in (M, g_w) vanishes.*

$f \rightsquigarrow (b, f)$

$F \hat{=} F^b$

Proof of Proposition 2.5.2:
"(a)": If we view $U, V \in \mathscr{X}(F^b)$ as vector fields on F, then $\tilde{U} := (\pi^F)^* U, \tilde{V} := (\pi^F)^* V \in \mathscr{X}(M)$ are extensions of U, V. In other words $\tilde{U}$ is the extension with $\tilde{U}\big|_{(\tilde{b},f)} = U\big|_{(b,f)}$ for all $(\tilde{b}, f) \in M$. $(\pi^B)^* X$

Let us extend X to $X \in \mathscr{X}(B)$, and set $\tilde{X} := \cancel{(\pi^F)^* X} \in \mathscr{X}(M)$. Then $[\tilde{X}, \tilde{U}] = 0$ and $[\tilde{X}, \tilde{V}] = 0$. Let ∇ be the Levi-Civita connection for g_{gwp}. We calculate

$$\begin{aligned}
g_{\mathrm{gwp}}\big(\vec{\mathrm{II}}^{F^b}(U,V), X\big) &= g_{\mathrm{gwp}}\big(\nabla_{\tilde{U}} \tilde{V}, \tilde{X}\big) \\
&= \partial_{\tilde{U}} \underbrace{g_{\mathrm{gwp}}(\tilde{V}, \tilde{X})}_{=0} - g_{\mathrm{gwp}}\big(\tilde{V}, \nabla_{\tilde{U}} \tilde{X}\big) \\
&= 0 - g_{\mathrm{gwp}}\big(\tilde{V}, \nabla_{\tilde{X}} \tilde{U}\big),
\end{aligned}$$

where we used $0 = [\tilde{X}, \tilde{V}] = \nabla_{\tilde{X}} \tilde{U} - \nabla_{\tilde{U}} \tilde{X}$.

Because of $\vec{\mathrm{II}}^{F^b}(U,V) = \vec{\mathrm{II}}^{F^b}(V,U)$ we get

$$\begin{aligned}
2 g_{\mathrm{gwp}}\big(\vec{\mathrm{II}}^{F^b}(U,V), X\big) &= -g_{\mathrm{gwp}}\big(\tilde{V}, \nabla_{\tilde{X}} \tilde{U}\big) - g_{\mathrm{gwp}}\big(\tilde{U}, \nabla_{\tilde{X}} \tilde{V}\big) \\
&= -\partial_{\tilde{X}}\big(g_{\mathrm{gwp}}(\tilde{V}, \tilde{U})\big) = -(\partial_{\tilde{X}} \hat{g})(U,V)
\end{aligned}$$

$\hat{V}$ is constant in $B \Rightarrow [\tilde{X}, \tilde{V}] = 0 = \mathcal{L}_{\tilde{X}} \tilde{V}$

$\partial_{\tilde{X}}(g_{\mathrm{gwp}}(\tilde{V}, \tilde{u})) = \mathcal{L}_X (g_{\mathrm{wp}} \ldots)$

$$\partial_{\tilde{X}}\left(\hat{g}\left(\hat{U},\hat{V}\right)\right) = \mathcal{L}_{\tilde{X}}\left(\hat{g}\left(\hat{U},\hat{V}\right)\right)$$

$$= \left(\mathcal{L}_{\tilde{X}}\hat{g}\right)\left(\hat{U},\hat{V}\right) + \hat{g}\left(\overbrace{\mathcal{L}_{\tilde{X}}\hat{U}}^{=0},\hat{V}\right)$$

$$+\ \hat{g}\left(\hat{U},\underbrace{\mathcal{L}_{\tilde{X}}\hat{V}}_{=0}\right)$$

$$= \left(\partial_{\tilde{X}}\hat{g}\right)\left(\hat{U},\hat{V}\right)$$

"(b)": This proof can be carried out similarly for $X, Y \in \mathscr{X}(B^f)$ and $U \in \mathrm{T}_f F$. One gets

$$2g_{\mathrm{gwp}}\big(\vec{\mathrm{II}}^{B^f}(X,Y),U\big) = -(\partial_{\tilde{U}}\check{g})(X,Y),$$

and $(\partial_{\tilde{U}}\check{g}) = 0$ as $\check{g}$ does not depend on f. ■

Remark 2.5.4. The above proposition says, roughly speaking, that the $-2\vec{\mathrm{II}}$ is the derivative of the semi-Riemannian metric on the submanifold F^b in normal directions. In fact one can show the following variation formula for the metric.

Let M be a semi-Riemannian submanifold $\cancel{M}$ of a semi-Riemannian manifold $(\overline{M},\overline{g})$ with vector-valued second fundamental form $\vec{\mathrm{II}}$. Let $f_t : \overline{M} \to \overline{M}$ be a family of diffeomorphisms, smoothly depending on $t \in (-\epsilon,\epsilon)$, and $f_0 = \mathrm{id}$. We define the **variation vector field** $\mathcal{V}\big|_p := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f_t(p)$ for $p \in M$. We assume further that $\mathcal{V} \in \Gamma(\mathrm{N}M)$. Then

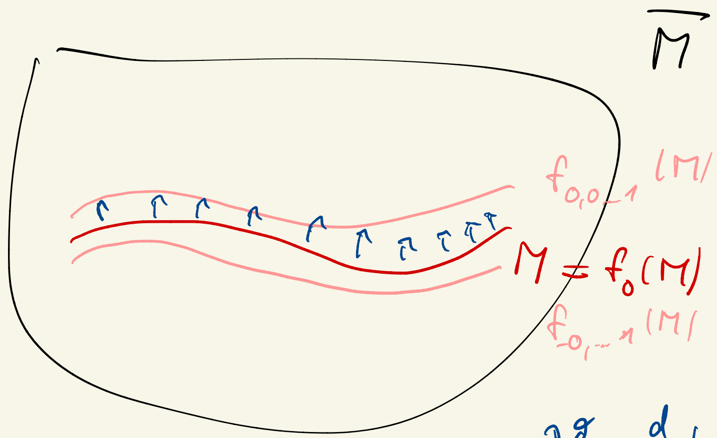
$g(\kappa, \vec{\mathrm{II}})$ $\partial_X \hat{g}$

$$g\big(\mathcal{V},\vec{\mathrm{II}}\big) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big((f_t^* g)_{\mathrm{T}M\times\mathrm{T}M}\big).$$

This specializes to the proposition by considering $\overline{M} = B \times F$, $M = \{b\} \times F$, $\mathcal{V} = \cancel{X}$. This result can be proved by slight extending arguments in the proof of the variational formula for the area.

$t \mapsto c(t)$ $\dot{c}(0) = X|_{c(0)}$

f_t is the flow of X



$f_t|_M$

$f_t(M) \subseteq M$

$$V_p = \frac{d}{dt}\Big|_{t=0} f_t(p)$$

$p \in M$

$X \in \mathscr{X}(\cancel{F}\,B) \Rightarrow (\pi^B)^*(X)$ ~~vertical~~ horizontal

Definition 2.5.5. *A vector field $X \in \mathscr{X}(B \times F)$ is called* **vertical**, *if for any $x = (b, f) \in M$ we have $X|_x \in \{0\} \times \mathrm{T}_f F$. A vector field $X \in \mathscr{X}(B \times F)$ is called* **horizontal**, *if for any $x = (b, f) \in M$ we have $X|_x \in \mathrm{T}_b B \times \{0\}$. Thus any $X \in \mathscr{X}(B \times F)$ can be written uniquely as $X = X_{\mathrm{hor}} + X_{\mathrm{ver}}$, where X_{hor} is horizontal, called the* **horizontal part** *of X, and where X_{ver} is vertical, called the* **vertical part** *of X.*

The Gauß and the Codazzi formula implies: X, Y, Z horizontal ; U, V, W vertical

Corollary 2.5.6. *With the above notation we have for horizontal vector fields $X, Y, Z \in \mathscr{X}(M)$*

$$R(X, Y)Z = \check{R}(X, Y)Z. \qquad \blacksquare$$

of (M, g_{wp}) of $(B, \check{g}) = (B^f, \check{g})$ $B^f = \{f\} \times B$

Definition 2.5.7. *A* **warped product metric** *on $M = B \times F$ is a generalized warped product metric $g_w = \check{g} + \hat{g}$ with the notation as above, such that there is a semi-Riemannian metric g^F on F and a smooth positive function $w \in \mathcal{C}^\infty(B, \mathbb{R}_{>0})$ such that $\hat{g}^b = w^2 g^F$. We thus have*

$$g_w = \check{g} + \underbrace{w^2 g^F}_{\hat{g}}.$$

In this special case we obtain

$w^2 g^F(U,V)$

$$g_w\big(\vec{\mathrm{II}}^{F^b}(U, V), X\big) = -(\partial_X w) w\, g^F(U, V) = -\underbrace{(\partial_X \log w)}_{\frac{\partial_X w}{w}}\, \hat{g}(U, V),$$

$$" -\frac{1}{2}\partial_X \hat{g}$$

or equivalently

$$\vec{\mathrm{II}}^{F^b}(U, V) = -\hat{g}(U, V)\,(\check{\mathrm{grad}} \log w). \quad \rightarrow \frac{1}{w}\,\mathrm{grad}\, w$$

Note that for a function $f \in \mathcal{C}^\infty(B)$ the g_w-gradient of $f \circ \pi^B$ coincides with the pullback of the $\check{g}$-gradient $\check{\mathrm{grad}} f \in \mathscr{X}(B) \subset \mathscr{X}(M)$. Hence we may also write grad instead of $\check{\mathrm{grad}}$.

Gauß

$$\langle R(X,Y)Z, \tilde{X} \rangle = \langle \check{R}(X,Y)Z, \tilde{X} \rangle + \langle \underbrace{\vec{II}(X,Z)}_{=0}, \vec{II}(Y,\tilde{X}) \rangle$$

$\tilde{X}$ horiz.

$$- \langle \underbrace{\vec{II}(Y,Z)}_{=0}, \vec{II}(X,\tilde{X}) \rangle$$

$B^f \subset M = B \times F$

$$\Rightarrow \pi^{\text{hor}}(R(X,Y)Z) = \check{R}(X,Y)Z$$

horizontal

<u>Codazzi</u> V vertical, normal to B^f

$$\langle R(X,Y)Z, V \rangle = \pm \Big(\underbrace{(\nabla_X II)}_{=0}(Y,Z) - \underbrace{(\nabla_Y II)}_{=0}(X,Z) \Big) = 0$$

$$\pi^{\text{ver}}(R(X,Y)Z) = 0$$

vertical

Example 2.5.8 ((Hyper-)surfaces of revolution)**.** We assume that we have a smooth function $f\colon (a,b) \to \mathbb{R}_{>0}$. We consider the curve $c(t) := (t, f(t))$ and the submanifold

$$M_{\mathrm{rot}} := \{(t,x)^\top \in (a,b) \times \mathbb{R}^n \mid f(t) = \|x\|\} \subset \mathbb{R}^{n+1}.$$

Riemannian or Lorentzian
Last Lecture

Example 2.5.9 (Robertson-Walker spacetimes)**.**
A **Robertson-Walker spacetime** is a manifold $M = (a,b) \times F$ with the Lorentzian metric $-\mathrm{d}t \otimes \mathrm{d}t + w(t)^2 \hat{g}$ where $(F, \hat{g})$ is Riemannian manifold of constant sectional curvature.

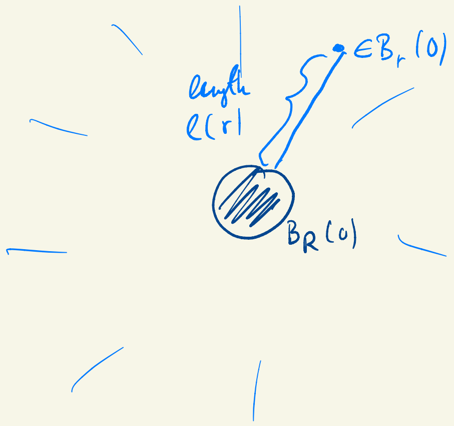
Example 2.5.10 (Rotationally symmetric Riemannian metrics)**.** Let g be a Riemannian metric on $\mathbb{R}^n \setminus B_R(0)$ which is invariant under the action of $O(n)$ on $\mathbb{R}^n \setminus B_R(0)$. We define for $r \geq R$ let $\ell(r)$ be the length of the straigth line from $(R, 0, \ldots)^\top$ to $(r, 0, \ldots)^\top$, and let $\ell_\infty := \lim_{r\to\infty} \ell(r)$. Then we define

$$\varphi : \mathbb{R}^n \setminus B_R(0) \to \overbrace{S^{n-1}}^{F} \times \overbrace{(0, \ell_\infty)}^{B}, \quad x \mapsto \left(\frac{x}{\|x\|}, \ell(\|x\|)\right)$$

Then $(\varphi^{-1})^* g$ is a Riemannian warped product metric on $S^{n-1} \times (0, \ell_\infty)$ given by

$$\mathrm{d}t \otimes \mathrm{d}t + w(t)^2 g^{S^{n-1}}$$

where $g^{S^{n-1}}$ is the standard metric on S^{n-1}.



length
ℓ(r)
∈B_r(0)
B_R(0)

Exercise 2.5.11. *With the above notation calculate for vertical vector fields $U, V, W \in \mathscr{X}(M)$*

$$R(U,V)W = \hat{R}(U,V)W - \frac{\check{g}(\check{\operatorname{grad}} w, \check{\operatorname{grad}} w)}{w^2}\big(\hat{g}(V,W)U - \hat{g}(U,W)V\big).$$

Hints: you have to consider both the horizontal and the vertical part of this equation. Consider the subsection on homotheties.

$w : B \to \mathbb{R}_{>0}$ warping function

For (M, g_w)

Lemma 2.5.12. *For $X \in \mathscr{X}(B)$ and $V \in \mathscr{X}(F)$ we have $\nabla_X V = \nabla_V X = (\partial_X \log w)V$.*

$B^f = B \times \{f\} \subset M = B \times F$

Proof: As X acts on functions by derivation in the B-direction, and V acts by derivation in the F direction, Schwarz's theorem implies $[X,V] = 0$ in the sense of commutator of vector fields on M. This implies $\nabla_V X = \nabla_X V$. For any horizontal vector field Y we calculate

$$g_w(\nabla_X V, Y) = \partial_X \overbrace{g_w(V,Y)}^{=0} - g_w(V, \nabla_X Y) = -g_w\big(V, \overbrace{\vec{\mathrm{II}}^B(X,Y)}^{=0}\big) = 0,$$

thus $\nabla_X V$ is vertical. For any vertical vector field U we calculate

$F^b = \{b\} \times F \subset B \times F$

$$\begin{aligned} g_w(\nabla_V X, U) &= \partial_V \overbrace{g_w(X,U)}^{=0} - g_w(X, \nabla_V U) = -g_w\big(X, \vec{\mathrm{II}}^F(V,U)\big) \\ &= (\partial_X \log w)\, g_w(V,U), \end{aligned}$$

and this implies the statement. ∎

For the following theorem note, that similar to the gradient, the pull-back of the Hessian of a function $f \in \mathcal{C}^\infty(B)$ to W coincides with the Hessian of $f \circ \pi^B$. Thus we have $\check{\nabla}^2 f = \nabla^2 f$.

Theorem 2.5.13 (Riemann curvature tensor for warped products)**.** *Let g_w be a warped product on $B \times F$ with the notation from above. Let X, Y, Z be horizontal and let U, V, W be vertical vector fields on M. Then*

$$\begin{aligned}
R(X,Y)Z &= \check{R}(X,Y)Z && (2.5.1)\\
R(X,U)Y &= \frac{(\nabla^2 w)(X,Y)}{w} U \\
&= \big((\nabla^2 \log w)(X,Y) + (\partial_X \log w)(\partial_Y \log w)\big) U && (2.5.2)\\
R(U,V)X &= 0 && (2.5.3)\\
R(X,Y)U &= 0 && (2.5.4)\\
R(X,U)V &= -\frac{g_w(U,V)}{w} \nabla_X (\operatorname{grad} w) && (2.5.5)\\
R(U,V)W &= \hat{R}(U,V)W \\
&\quad - \frac{g(\operatorname{grad} w, \operatorname{grad} w)}{w^2} \big(\hat{g}(V,W)U - \hat{g}(U,W)V\big). && (2.5.6)
\end{aligned}$$

Proof: At several places we will use $\partial_X \log w = \frac{\partial_X w}{w}$.
"(2.5.1)": See Corollary 2.5.6.
"(2.5.2)": We give two formulas, as both are helpful in applications. The additional transformations are given in blue.

$$\begin{aligned}
\nabla_X \nabla_U Y &\overset{\text{Lem. 2.5.12}}{=} \nabla_X \left((\partial_Y \log w) U\right) \\
&\overset{\text{Lem.}}{=} \left(\partial_X \partial_Y \log w + (\partial_Y \log w)(\partial_X \log w)\right) U \\
&= \frac{\partial_X \partial_Y w}{w} U
\end{aligned}$$

As we know that $\vec{\mathrm{II}}^{B^f} = 0$ for all $f \in F$, we know that $\nabla_X Y = \check{\nabla}_X Y$.

$$\begin{aligned}
\nabla_U \nabla_X Y = \nabla_U \check{\nabla}_X Y &= \left(\partial_{\check{\nabla}_X Y} \log w\right) U \\
&= \frac{\partial_{\check{\nabla}_X Y} w}{w} U.
\end{aligned}$$

$$R(X,U)Y = \nabla_X \nabla_U Y - \nabla_U \nabla_X Y - \nabla_{[X,U]} Y$$

$$\nabla_X V = \left(\nabla_X (\log w)\right) V$$

$$\shortparallel$$

$$\nabla_V X$$

$$\partial_x \partial_y \log w = \partial_x \frac{\partial_y w}{w} = \frac{\partial_x \partial_y w}{w} - \frac{(\partial_y w)(\partial_x w)}{w^2}$$

$$= \frac{\partial_x \partial_y w}{w} - \left(\partial_x (\log w)\right)\left(\partial_y (\log w)\right)$$

R(U,V)X

Because of $[X,U] = 0$ this provides the requested formulae.
"(2.5.3)": We get

$$\begin{aligned}\nabla_U\nabla_V X &\overset{\text{Lem. 2.5.12}}{=} \nabla_U\left((\partial_X \log w)V\right)\\ &= \underbrace{(\partial_U\partial_X \log w)}_{=0} V + (\partial_X \log w)\nabla_U V\end{aligned}$$

and thus using again Lemma 2.5.12 for $\nabla_{[U,V]}X$

$$R(U,V)X = (\partial_X \log w)\left(\nabla_U V - \cancel{\nabla_V U -} [U,V]\right) = 0.$$

"(2.5.4)": $R(X,Y)U$ is horizontal because of

$$g(R(X,Y)U,V) = g(R(U,V)X,Y) \overset{(2.5.3)}{=} 0.$$

On the other hand

$$g(R(X,Y)U,Z) = -g(R(X,Y)Z,U) \overset{(2.5.1)}{=} 0.$$

"(2.5.5)": $R(X,U)V$ is horizontal because of

$$g\left(R(X,U)V,W\right) = g\left(R(V,W)X,U\right) \overset{(2.5.3)}{=} 0.$$

For the horizontal part we calculate

$$\begin{aligned}g\left(R(X,U)V,Y\right) &= -g\left(R(X,U)Y,V\right)\\ &\overset{(2.5.2)}{=} -g\left(\frac{(\nabla^2 w)(X,Y)}{w}U,V\right)\\ &\overset{(*)}{=} -\frac{g_w(U,V)}{w}g\left(\nabla_X(\operatorname{grad} w),Y\right),\end{aligned}$$

where we used at the transformation $(*)$ the remark after Defini-