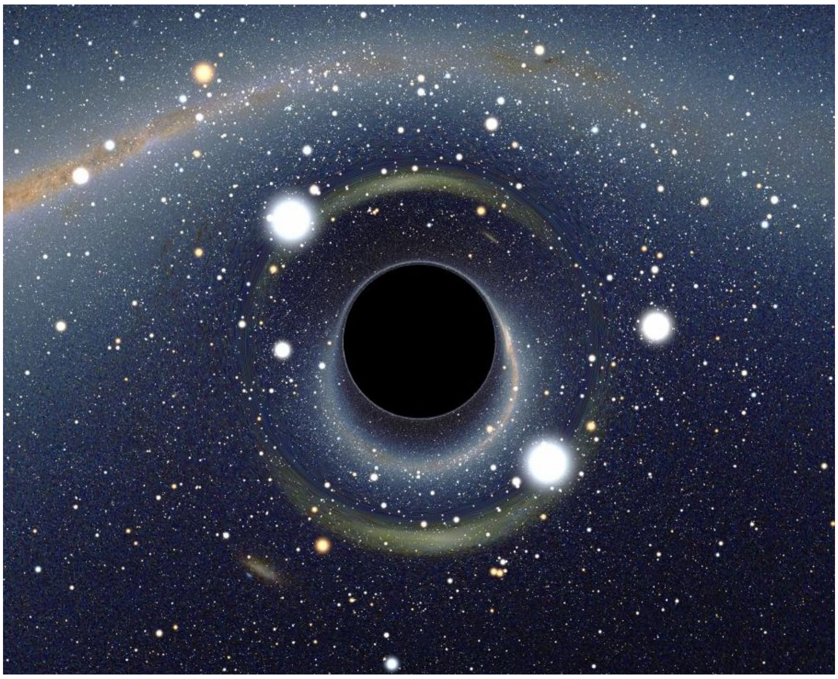
Differential Geometry II
Lorentzian Geometry
Lecture Notes



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Presentation Version

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Recapitulation before the lecture on 03.05.:

Additions at the end of Secion 2.1: Note that for functions $f \in \mathcal{C}^\infty(M)$ we have according to Definition 2.1.14

t

Semi-Riem.

grad f = (df)#
= b^{-1}
X ↦^b g(X, ·)

$$\nabla f = \partial f = \mathrm{d}f \in \Gamma(\mathrm{T}^*M).$$

We can derivate once again according to Definition 2.1.14 and get $\nabla^2 f := \nabla(\nabla f)$, and we obtain

$$\begin{aligned}(\nabla^2 f)(X,Y) &= \big(\nabla(\mathrm{d}f)\big)(X,Y) = \big(\nabla_X(\mathrm{d}f)\big)(Y)\\ &= \partial_X(\mathrm{d}f(Y)) - \mathrm{d}f(\nabla_X Y) = \partial_X\partial_Y f - \partial_{\nabla_X Y} f\end{aligned}$$

$\nabla : \Gamma(T^{0,1}M) \to \Gamma(T^{0,2}M)$

The tensor $\nabla^2 f \in \Gamma(\mathrm{T}^{0,2}M)$ is called the **Hessian** of f, and we also write it as $\operatorname{Hess} f$. The fact that ∇ is torsionfree is equivalent to $(\nabla^2 f)(X,Y) = (\nabla^2 f)(Y,X)$ for all $X, Y \in \mathscr{X}(M)$.

$(\nabla g) = 0$

Remarks.

(a) For a vector field $X \in \mathscr{X}(M)$ and a function $f \in \mathcal{C}^\infty(M)$, we can show $\nabla_X \operatorname{grad} f = (\nabla_X(\mathrm{d}f))^\#$. In order to show this, we check for $Y, Z \in \mathscr{X}(M)$

$$\begin{aligned}(\nabla_X Y)^\flat(Z) &= g(\nabla_X Y, Z) = \partial_X(g(Y,Z)) - g(Y, \nabla_X Z)\\ &= \partial_X(Y^\flat(Z)) - Y^\flat(\nabla_X Z) = \big(\nabla_X(Y^\flat)\big)(Z),\end{aligned}$$

which proves $(\nabla_X Y)^\flat = \nabla_X(Y^\flat)$ and which we apply to $Y = \operatorname{grad} f$ to get the statement.

(b) We have $(\nabla^2 f)(X,Y) = g(\nabla_X(\operatorname{grad} f), Y)$. In order to show this we calculate

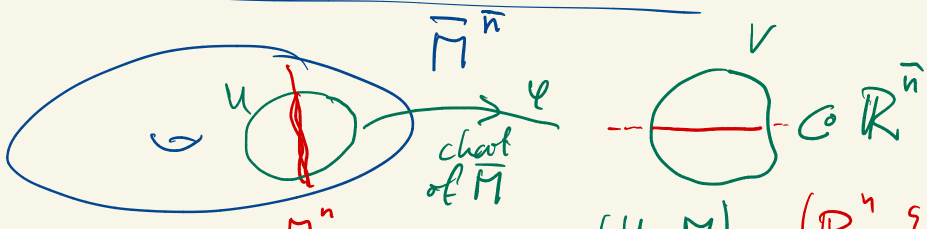
$$\begin{aligned}(\nabla^2 f)(X,Y) &= (\nabla_X \mathrm{d}f)(Y) = \big(\nabla_X(\operatorname{grad} f)^\flat\big)(Y)\\ &= \big(\nabla_X \operatorname{grad} f\big)^\flat(Y)) = g(\nabla_X \operatorname{grad} f, Y)\end{aligned}$$

$$(\nabla^2 f)(X,Y) = \partial_X \partial_Y f - \partial_{\nabla_X Y} f \qquad +$$

$$(\nabla^2 f)(Y,X) = \partial_Y \partial_X f - \partial_{\nabla_Y X} f \qquad -$$

$$(\nabla^2 f)(X,Y) - (\nabla^2 f)(Y,X) = \partial_{[X,Y]} f - \partial_{\nabla_X Y - \nabla_Y X} f$$

$$= 0 \quad (\text{torsionfree } \nabla!)$$



$$\varphi(U \cap M) = (\mathbb{R}^n \times \{0\}) \cap V$$

$M \subset \overline{M}$ is a submfd if such a φ exists.

on $\mathbb{R}^n$ $\nabla^2 f(\partial_i, \partial_j) = \frac{\partial^2 f}{\partial x^i \partial x^j}$

The **Laplacian** Δf is defined as

$$\Delta f\big|_x := -\sum_{i=1}^{\dim M} \epsilon_i (\nabla^2 f)(e_i, e_i),$$

where $(e_1, \ldots, e_{\dim M})$ is a generalized orthonormal basis of $(\mathrm{T}_x M, g_x)$. The linear map $\mathcal{C}^{k+2}(M) \to \mathcal{C}^k(M)$, $f \mapsto \Delta f$ for some $k \in \mathbb{N}_0 \cup \infty$ is called the **Laplace operator** or the **Laplace-Beltrami operator** of (M, g).
Attention: Sign of Δ

Last lecture: We studied submanifolds $\iota : M \hookrightarrow \overline{M}$. Here $\overline{M}$ carries a semi-Riemannian metric $\overline{g}$, and M is such that $\iota^* \overline{g}$ is a semi-Riemannian metric on M.

$\dim M = n = \overline{n} - 1$, $\overline{n} = \dim \overline{M}$

2.3 Semi-Riemannian hypersurfaces

We now specialize to the case that M is **hypersurfaces** in $\overline{M}$, i. e., $\dim M = \dim \overline{M} - 1$. Again M is called a **semi-Riemannian hypersurface** if $g := \overline{g}_{\mathrm{T}M \otimes \mathrm{T}M} = \iota^* \overline{g}$ is non-degenerate. In this case the normal bundle $\mathrm{N}M \to M$ has rank 1. If M is connected this implies that $\overline{g}|_{\mathrm{N}M \otimes \mathrm{N}M}$ is either positive definite (then we say the hypersurface has **sign** $\mathrm{sgn}(M) := +1$, or it is negative definite (then the sign is $\mathrm{sgn}(M) = -1$).

Example 2.3.1. Assume t is a regular value of the smooth function. $f : \overline{M} \to \mathbb{R}$. Then $M := f^{-1}(t)$ is a submanifold, in fact a hypersurface, and $\mathrm{T}_p M = \ker \mathrm{d}_p f = \left(\operatorname{grad} f\big|_p\right)^\perp$ for $p \in M$. If $\mathrm{T}_p M$ is a non-degenerate subspace, we can take a generalized orthonormal basis of $\mathrm{T}_p M$ and complete it to a generalized orthonormal basis of $\mathrm{T}_p \overline{M}$ by joining $\sqrt{|\langle \operatorname{grad} f|_p, \operatorname{grad} f|_p \rangle|}^{\,-1} \operatorname{grad} f\big|_p$. Thus if $\mathrm{T}_p M$ is non-

$\operatorname{grad} f = (df)^\#$

degenerate, grad f is non-zero and non-lightlike. Conversely if $\cancel{\mathrm{T}_p M}$ grad f is non-zero and non-lightlike, then $\mathrm{T}_p M = \left(\operatorname{grad} f|_p\right)^\perp$ is non-degenerate. Thus if M is given as $f^{-1}(t)$ for a regular value t, then M is a semi-Riemannian hypersurface if and only if grad f is nowhere lightlike and this holds if and only if df is nowhere lightlike.

Definition 2.3.2. *A* **unit normal field** *of a semi-Riemannian ~~submanifold~~ hypersurface M in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is a section $\nu \in \Gamma(\mathrm{N}M)$ with $\overline{g}(\nu,\nu) \in \{-1,+1\}$.*
We say M is **co-orientable** *if a unit normal field exists, and the choice of such a unit normal field is called a* **co-orientation***.*

Example 2.3.3. The hypersurface $f^{-1}(t)$ in the preceding example is co-orientable.
A unit normal field is given by $\nu := \sqrt{|\langle \operatorname{grad} f|_p, \operatorname{grad} f|_p\rangle|}^{-1} \operatorname{grad} f|_p$.

Definition 2.3.4. *Assume that ν is a unit normal field on the hypersurface $M \subset \overline{M}$. Then we define for $X \in \mathscr{X}(M)$ the* **shape operator** *$S(X) := -\overline{\nabla}_X \nu$.*

We calulate $0 = \partial_X \big(\overbrace{\overline{g}(\nu,\nu)}^{\pm 1}\big) = 2g(\overline{\nabla}_X \nu, \nu) = -2g(S(X),\nu)$, and thus $S(X) \in \mathscr{X}(M)$. The map $X \mapsto S(X)$ is $\mathcal{C}^\infty(M)$-linear and thus given by a tensor $S \in \Gamma(\mathrm{T}^*M \otimes \mathrm{T}M) = \Gamma(\mathrm{End}(\mathrm{T}M))$.

Lemma 2.3.5. *For all $X, Y \in \mathscr{X}(M)$ we have*

$$g(S(X),Y) = \overline{g}(\vec{\mathrm{II}}(X,Y),\nu)$$

Proof:

$$\begin{aligned} 0 &= \partial_X \overbrace{\overline{g}(\nu,Y)}^{\equiv 0} = \overline{g}(\overline{\nabla}_X \nu, Y) + \overline{g}(\nu, \overline{\nabla}_X Y) \\ &= -\overline{g}(S(X),Y) + \overline{g}(\nu, \vec{\mathrm{II}}(X,Y)). \end{aligned}$$

■

Because of the lemma, it is convenient to define the (scalar-valued) **second fundamental form** as $\mathrm{II}(X,Y) = \overline{g}(\vec{\mathrm{II}}(X,Y),\nu)$. Note that this implies the sign $\vec{\mathrm{II}}(X,Y) = \mathrm{sgn}(M)\mathrm{II}(X,Y)\nu$.

Minkowsk

Example 2.3.6 (Hyperbolic space)**.** Let again $\langle\!\langle\,\bullet,\bullet\,\rangle\!\rangle$ be the ~~Lorentzian~~ standard scalar product of $\mathbb{R}^{m,1}$. Let $\perp\!\!\!\perp$ denote orthogonality with respect to $\langle\!\langle\,\bullet,\bullet\,\rangle\!\rangle$, e. g., $x^{\perp\!\!\!\perp}$ or $V^{\perp\!\!\!\perp}$ is the orthogonal space for $\langle\!\langle\,\bullet,\bullet\,\rangle\!\rangle$. We define

$H^m = f^{-1}(-1) \cap \{x^0 > 0\}$ $\qquad f = \frac{1}{2}\langle\!\langle x,x\rangle\!\rangle$

$$H^m := \left\{x = (x^0, x^1, \ldots, x^m)^\top \in \mathbb{R}^{m,1} \mid \langle\!\langle x,x\rangle\!\rangle = -1 \text{ and } x^0 > 0\right\}.$$

A unit normal field is given by the identity $x \mapsto \nu(x) = x$, as we have $\langle\!\langle x,x\rangle\!\rangle = -1$ and $\mathrm{T}_x H^m = x^{\perp\!\!\!\perp}$. (More on Exercise sheet 3). This is a model for hyperbolic space.

$T_x H^m$ $\qquad$ $\operatorname{grad} f = x$

Theorem 2.3.7 (Gauß formula for hypersurfaces)**.** *Let $M \subset \overline{M}$ be a semi-Riemannian hypersurface with unit normal field ν, $\epsilon := \langle \nu,\nu\rangle \in \{-1,+1\}$ its sign.*

$\pi^T(\overline{R}(X,Y)Z)$

(i) *For $X,Y,Z \in \mathrm{T}_p M$ we have*

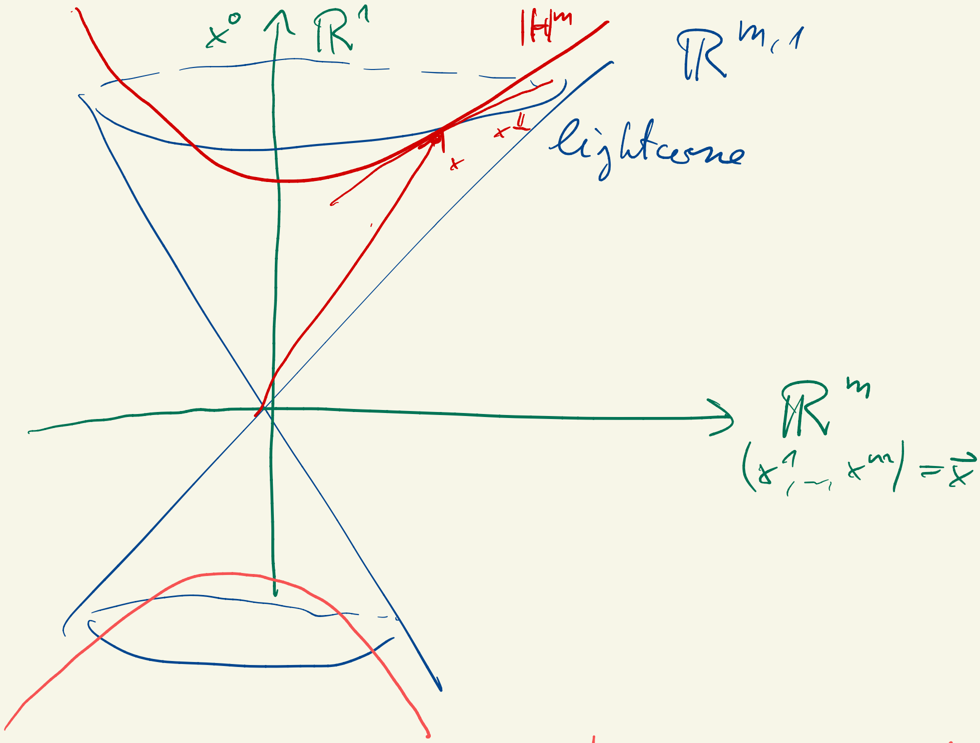
$$R(X,Y)Z = \overline{R}(X,Y)Z + \epsilon\left(\langle S(Y),Z\rangle S(X) - \langle S(X),Z\rangle S(Y)\right).$$

(ii) *For any $E = \mathrm{span}\{X,Y\} \in G_2(M,g)$ we have*

$$\begin{aligned}\sec(E) &= \overline{\sec}(E) + \varepsilon\frac{\langle S(X),X\rangle\langle S(Y),Y\rangle - \langle S(X),Y\rangle^2}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}\\ &= \overline{\sec}(E) + \epsilon \det S|_E\end{aligned}$$

■

$$\sec(E) = \frac{g(R(X,Y)Y,X)}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$$



$$-\mathbb{H}^m = \{ x \in \mathbb{R}^{m,1} \mid \langle\langle x, x \rangle\rangle = -1,\ x^0 < 0 \}$$

Example 2.3.8 (De Sitter spacetime)**.** An important example of a Lorentzian manifold is **de Sitter spacetime** which is the pseudosphere defined for some $r > 0$ as

sign$(\mathbb{S}^{n,1}(r))$
$= +1$

$$\mathbb{S}^{n,1}(r) := \left\{ x \in \mathbb{R}^{n+1,1} \;\middle|\; \langle\!\langle x, x \rangle\!\rangle = r^2 \right\}.$$

The de Sitter space $\mathbb{S}^{n,1}(r)$ is diffeomorphic to $\mathbb{S}^n \times \mathbb{R}$, see Exercise Sheet 1, Exercise 2, thus for $n \geq 2$ it is simply connected. This example is treated in Exercise Sheet 3, Exercise 1 – up to scaling by r.

Example 2.3.9 (Anti-de Sitter spacetime)**.** Let us consider for some $r > 0$ the pseudohyperbolic space

sign$(\mathbb{H}^{n,1}(r))$
$= -1$

$$\mathbb{H}^{n,1}(r) := \left\{ x \in \mathbb{R}^{n,2} \;\middle|\; \langle\!\langle x, x \rangle\!\rangle_{n,2} = -r^2 \right\}.$$

The space $\mathbb{H}^{n,1}(r)$ is diffeomorphic to $\mathbb{R}^n \times \mathbb{S}^1$, see Exercise Sheet 1, Exercise 2, thus it is never simply connected. Its universal covering $\widetilde{\mathbb{H}}^{n,1}(r)$, which is diffeomorphic to $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ is called **anti-de Sitter spacetime**. This example is treated in Exercise Sheet 3, Exercise 1 – up to scaling by r.

Physical Interpretation 2.3.10. *The Minkowski space $\mathbb{R}^{n,1}$, de Sitter spacetime $\mathbb{S}^{n,1}$ and anti-de Sitter spacetime $\widetilde{\mathbb{H}}^{n,1}(r)$ are vaccum solutions of the Einstein equations for vanishing, positive and negative cosmological constant.*

Note the normal bundle also carries a natural connection, namely for $X \in \mathscr{X}(M)$ and $V \in \Gamma(\mathrm{N}M)$ we define

$$\overline{\nabla}^{\perp}_X V := \pi^{\mathrm{N}}(\overline{\nabla}_X V).$$

$\nabla_X Y$

$Y \in \Gamma(TM)$
$Y \in \Gamma(T\overline{M}|M)$

$\overline{\nabla}^{\perp\!\!\perp}$

$\vec{\mathrm{II}}_{|p}: T_pM \times T_pM \to N_pM$

We then define

$$(\nabla_Z \vec{\mathrm{II}})(X,Y) := \underbrace{\overline{\nabla}^{\perp\!\!\perp}_Z\big(\vec{\mathrm{II}}(X,Y)\big)}_{\in \Gamma(NM)} - \vec{\mathrm{II}}(\nabla_Z X, Y) - \vec{\mathrm{II}}(X, \nabla_Z Y).$$

Theorem 2.3.11 (Codazzi)**.** *For vector fields $X, Y, Z \in \mathscr{X}(M)$ on a semi-Riemannian submanifold in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ we have:*

$$\pi^{\mathrm{N}}(\overline{R}(X,Y)Z) = (\nabla_X \vec{\mathrm{II}})(Y,Z) - (\nabla_Y \vec{\mathrm{II}})(X,Z).$$

Proof: By defintion of $\overline{R}$ we have: $X, Y, Z \in \mathfrak{X}(M) \in \Gamma(T\overline{M}_{|M})$

$$\pi^{\mathrm{N}}\left(\overline{R}(X,Y)Z\right) = \pi^{\mathrm{N}}\big(\overline{\nabla}_X \underbrace{\overline{\nabla}_Y Z}_{\in \Gamma(T\overline{M}_{|M})} - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z\big).$$

We calculate for each summand:

$$\begin{aligned}
\pi^{\mathrm{N}}\left(\overline{\nabla}_X \overline{\nabla}_Y Z\right) &= \pi^{\mathrm{N}}\Big(\overline{\nabla}_X\big(\nabla_Y Z + \vec{\mathrm{II}}(Y,Z)\big)\Big)\\
&= \vec{\mathrm{II}}\left(X, \nabla_Y Z\right) + \overline{\nabla}^{\perp\!\!\perp}_X\big(\vec{\mathrm{II}}(Y,Z)\big)\\
&= (\nabla_X \vec{\mathrm{II}})(Y,Z) + \vec{\mathrm{II}}\left(X, \nabla_Y Z\right) + \vec{\mathrm{II}}(\cancel{\nabla_X Y}, Z) + \vec{\mathrm{II}}(Y, \nabla_X Z),
\end{aligned}$$

and obviously the same holds when X and Y are permuted.

$$\pi^{\mathrm{N}}\left(\overline{\nabla}_{[X,Y]} Z\right) = \vec{\mathrm{II}}([X,Y], Z) = \vec{\mathrm{II}}(\cancel{\nabla_X Y, Z}) - \vec{\mathrm{II}}(\cancel{\nabla_Y X, Z})$$

We collect together and obtain

$$\pi^{\mathrm{N}}\left(\overline{R}(X,Y)Z\right) = (\nabla_X \vec{\mathrm{II}})(Y,Z) - (\nabla_Y \vec{\mathrm{II}})(X,Z)$$

as claimed. ∎

$\overline{\nabla}_Y Z = \nabla_Y Z + \vec{\mathrm{II}}(Y,Z)$

2.4 Homotheties

Definition 2.4.1. *Two semi-Riemannian metrics g and $\tilde{g}$ on a manifold M are* **homothetic**, *if there is a constant $t > 0$ such that $\tilde{g} = t^2 g$.*

Lemma 2.4.2. *We assume the above notation, and we indicate by a tilde (or no tilde) whether the following geometric objects are defined with respect to $\tilde{g}$ or with respect to g. Then for $X, Y, Z, W \in \mathfrak{X}(M)$ we have*

$$
\begin{array}{rcl}
\widetilde{\nabla}_X Y & = & \nabla_X Y \\
\widetilde{R}(X,Y)Z & = & R(X,Y)Z \\
\cancel{\tilde{g}(\widetilde{R}(X,Y)Z)} & = & t^2 \cancel{g(R(X,Y)Z)} \\
\widetilde{\sec}(E) & = & t^{-2} \sec(E) \\
\cancel{\widetilde{\mathrm{ric}}(X)} & = & \cancel{\mathrm{ric}(X)} \\
\widetilde{\mathrm{Ric}}(X) & = & t^{-2}\,\mathrm{Ric}(X) \\
\widetilde{\mathrm{RIC}}(\mathbb{R}X) & = & t^{-2}\,\mathrm{RIC}(\mathbb{R}X) \\
\widetilde{\mathrm{scal}} & = & t^{-2}\,\mathrm{scal}
\end{array}
$$

$\tilde{g}(\widetilde{R}(X,Y)Z,W)$ $g(R(X,Y)Z,W)$

$\widetilde{\mathrm{ric}}(X,Y)$ $\mathrm{ric}(X,Y)$

Proof: Consider the properties defining the Levi-Civita connection. They do not depend on the semi-Riemann metric, except Product Rule II. By passing from the metric g to $\tilde{g}$ both sides of Product Rule II are changed by a factor t^2, thus the product rule II for $\tilde{g}$ is also equivalent to the one for g. Thus the uniqueness statement in Theorem 2.1.2 implies $\widetilde{\nabla}_X Y = \nabla_X Y$.

The other transformation formulas now follow with elementary calculations. ■

2.5 Warped products

For smooth manifolds B and F we will consider their product manifold $M = B \times F$. Sometimes B will be considered as the **basis** of M and $\{x\} \times F$, $x \in B$ as its fibers. The projection to B resp. to F will be denoted by π^B resp. π^F. We assume that B carries a semi-Riemannian metric $\check{g}$ with associated Levi-Civita connection $\check{\nabla}$ and curvatures $\check{R}$, $\check{\sec}$, $\check{\mathrm{ric}}$, $\check{\mathrm{scal}}$,.... On F we do not consider one single metric, but a smooth family $\hat{g}^b \in \Gamma(\mathrm{T}^*F \odot \mathrm{T}^*F)$ is semi-Riemannian ~~manifolds~~ metric, depending smoothly on $b \in B$.

Definition 2.5.1. *On* $\mathrm{T}_{(b,f)}(B \times F) = \mathrm{T}_bB \oplus \mathrm{T}_fF$ *we define the inner product* $g^{\mathrm{gwp}}_{(b,f)} := \check{g}_b \oplus \hat{g}^b_f$, *i. e., for* $X_1, X_2 \in \mathrm{T}_bB$ *and* $Y_1, Y_2 \in \mathrm{T}_fF$ *we define*

$$g^{\mathrm{gwp}}_{(b,f)}\big((X_1,Y_1),(X,Y_2)\big) := \check{g}_b(X_1,X_2) + \hat{g}^b_f(Y_1,Y_2).$$

This defines a semi-Riemannian metric g_{gwp} *on* $M = B \times F$. *Metrics of this form will be called* **generalized warped product metrics**.

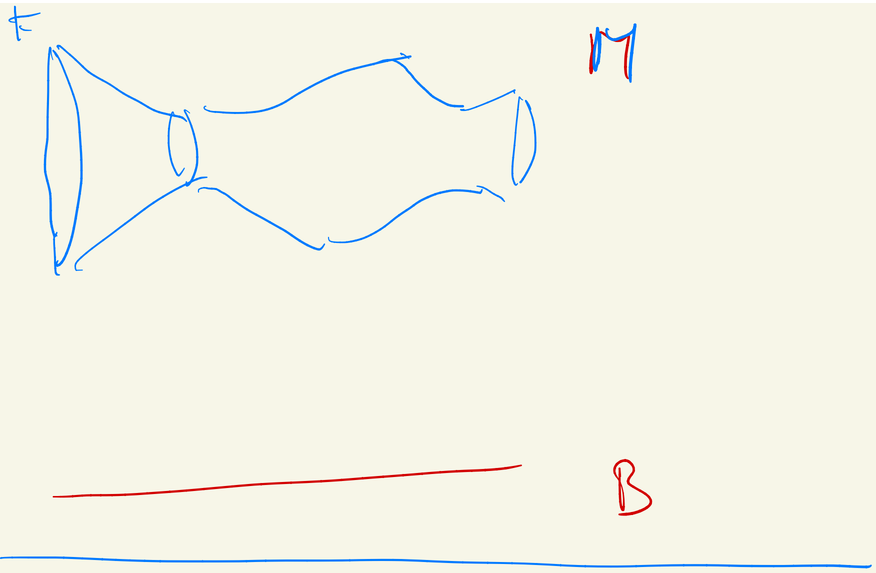
$(\pi^B)^*\check{g}\,,\ \hat{g} \in \Gamma(T^*M \odot T^*M)$

symmetric

Alternatively we could say

$$g^{\mathrm{gwp}}_{(b,f)} = \left(\pi^B\right)^* \check{g} + \hat{g}$$

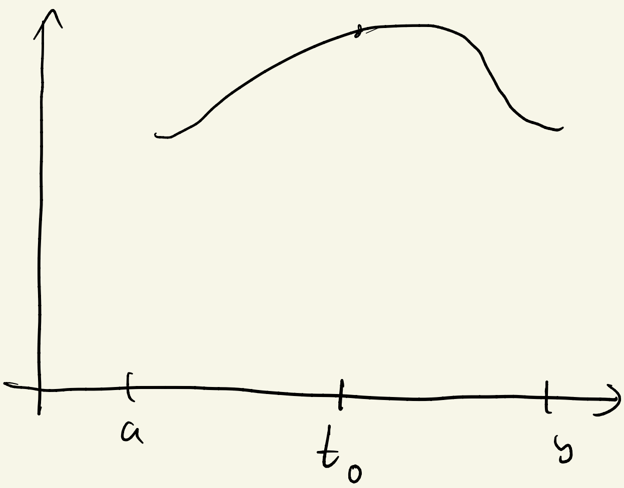
where $\hat{g} \in \Gamma(\mathrm{T}^*M \odot \mathrm{T}^*M)$ is defined by $\hat{g}_{(b,f)} = 0 + \hat{g}^b_f$. Note that we can view $\hat{g}$ as a function $B \to \Gamma(\mathrm{T}^*F \odot \mathrm{T}^*F)$ and the smooth dependense on b allows to define $\partial_X \hat{g} \in \Gamma(\mathrm{T}^*F \odot \mathrm{T}^*F)$ for any $X \in \mathrm{T}B$.



Example 2.5.8 **(Hyper)-Surfaces of Revolution**

$f : (a,b) \to \mathbb{R}_{>0}$ $\quad C^\infty \quad$ $t_0 \in (a,b)$

$c(t) = (t, f(t))$



$$\ell(t) := \int_{t_0}^{t} \sqrt{1 + f'(\tau)^2}\, d\tau$$

$$= \begin{cases} \mathcal{L}(c_{|[t_0,t]}) & t \geq t_0 \\ -\mathcal{L}(c_{|[t,t_0]}) & t < t_0 \end{cases}$$

$M_{rot} := \{ (t,x) \in (a,b) \times \mathbb{R}^n \mid f(t) = \|x\| \}$ $\quad \mathbb{R}^{n,1}$

g_{rot} induced Riem.-metric $\cancel{\mathbb{R}^{n+1}}$

$$M_{rot} \xleftarrow[\text{diffeo}]{\varphi} (\ell_0,\ell_1)\times S^{n-1}$$

$$(t,x) \longmapsto \left(\ell(t), \frac{x}{\|x\|}\right)$$

$$\left(\ell^{-1}(s), f\circ\ell^{-1}(s)\, y\right) \longmapsfrom (s, y)$$

$\ell : (a,b) \longrightarrow (\ell_0, \ell_1)$ diffeom.

$$\varphi^* g_{rot} = -ds^2 + (f\circ\ell^{-1})(s)^2 \, g^{S^{n-1}}$$

warped product

$B = (\ell_0, \ell_1)$, $F = S^{n-1}$

$\check{g} = -ds^2$ $\qquad$ $\hat{g}^b = (f\circ\ell^{-1})(b)^2 \, g^{S^{n-1}}$

end. $\qquad$ $b = s$