B Related vector fields

Definition B.1. *Let M and N be smooth manifolds, and $f : M \to N$ a smooth map. A vector $\overline{X} \in \mathscr{X}(N)$ is called* **f-related** *to $X \in \mathscr{X}(M)$ if $\mathrm{d}f \circ X = \overline{X} \circ f$, i. e., if the diagram*

$$\begin{tikzcd} \mathrm{T}M \arrow[r, "\mathrm{d}f"] & \mathrm{T}N \\ M \arrow[u, "X"] \arrow[r, "f"] \arrow[ur, phantom, "\circlearrowleft"] & N \arrow[u, "\overline{X}"'] \end{tikzcd}$$

commutes.

Lemma B.2. *Let M and N be smooth manifolds, and $f : M \to N$ a smooth map. If $\overline{X} \in \mathscr{X}(N)$ is f-related to $X \in \mathscr{X}(M)$ and if $\overline{Y} \in \mathscr{X}(N)$ is f-related to $Y \in \mathscr{X}(M)$, then $[\overline{X},\overline{Y}]$ is f-related to $[X,Y]$.*

Proof: For any $\varphi \in \mathcal{C}^\infty(N)$ we have

$$\partial_{\mathrm{d}f(X)}\varphi = \mathrm{d}\varphi(\mathrm{d}f(X)) = \mathrm{d}(\varphi \circ f)(X) = \partial_X(\varphi \circ f).$$

Thus we have $\partial_X(\varphi \circ f) = \partial_{\overline{X}\circ f}\varphi = (\partial_{\overline{X}}\varphi) \circ f$ and similarly $\partial_Y(\varphi \circ f) = (\partial_{\overline{Y}}\varphi) \circ f$ for all $\varphi \in \mathcal{C}^\infty(N)$. We now calculate

$$\begin{aligned} \partial_{\mathrm{d}f([X,Y])}\varphi &= \partial_{[X,Y]}(\varphi \circ f) \\ &= \partial_X\big(\partial_Y(\varphi \circ f)\big) - \partial_Y\big(\partial_X(\varphi \circ f)\big) \\ &= \partial_X\big((\partial_{\overline{Y}}\varphi) \circ f\big) - \partial_Y \text{bigl}((\partial_{\overline{X}}\varphi) \circ f\big) \\ &= \big(\partial_{\overline{X}}(\partial_{\overline{Y}}\varphi)\big) \circ f - \big(\partial_{\overline{Y}}(\partial_{\overline{X}}\varphi)\big) \circ f \\ &= (\partial_{[\overline{X},\overline{Y}]}\varphi) \circ f. \end{aligned}$$

$[\overline{X},\overline{Y}]_{f(p)} = d_pf([X,Y])_p$

As derivations are in bijection to vector fields, this implies $\mathrm{d}f([X,Y]) = [\overline{X},\overline{Y}] \circ f$, which is the statement of the lemma. ∎