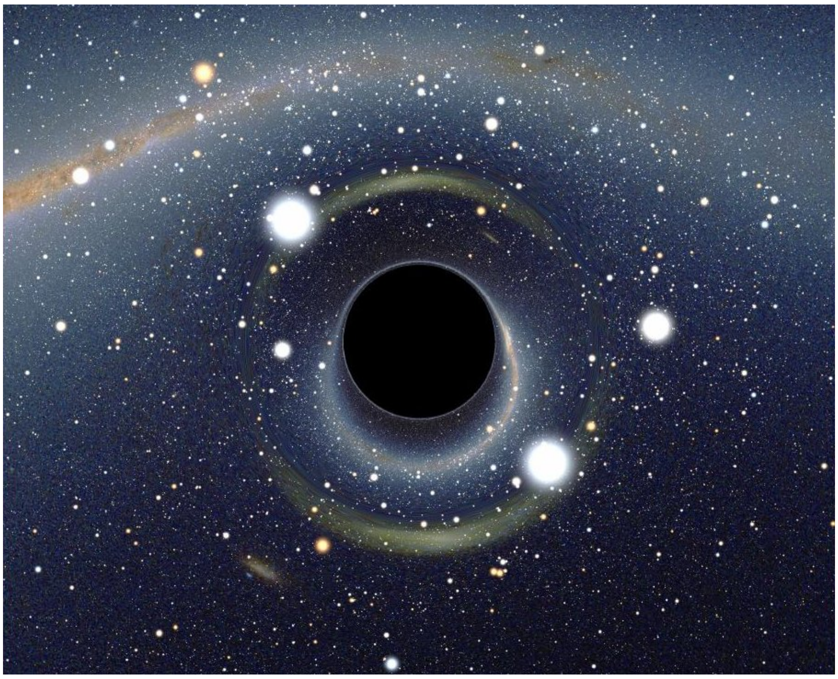
Differential Geometry II
Lorentzian Geometry

Lecture Notes



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Definition 2.1.16. *Let g be a non-degenerate symmetric form on a vector space V. A subspace U is called* **non-degenerate subspace** *of (V,g) if $g|_{U\times U}$ is non-degenerate. The space of all k-dimensional non-degenerate subspaces of (V,g) is called the* **Grassmannian of non-degenerate k-dimensional subspaces** *denoted as $G_k(V,g)$. Obviously $G_k(V,g)$ is an open subset of the space $G_k(V)$ of all k-dimensional subspaces of V.*

If (M,g) is a semi-Riemannian manifold, then we define (as a set)

$$E \subset G_k(M,g) = \dot{\bigcup_{p\in M}} \underbrace{G(\mathrm{T}_pM, g_p)}_{\in E}.$$

We equip $G_k(M,g)$ with the unique differential structure such that

$$\begin{array}{rcl} \underbrace{TM\times\dots\times TM}_{k\text{-times}} \supset \{\underbrace{(e_1,\dots,e_k)}_{\operatorname{span}=E} \mid p\in M, \underbrace{\operatorname{span}\{e_1,\dots,e_k\}}_{\substack{\shortparallel \\ E}} \in G_k(\mathrm{T}_pM, g_p)\} & \to & G_k(M,g) \\ (e_1,\dots,e_k) & \mapsto & \operatorname{span}\{e_1,\dots,e_k\} \end{array}$$

is a smooth map with surjective differential.

Recall that Lemma 1.1.12 tells us that a plane E is in $G_2(M,g)$, if and only if $g(X,X)g(Y,Y) - g(X,Y)^2 \neq 0$. for $\operatorname{span}\{X,Y\} = E$

Definition 2.1.17. *Let (M,g) be a semi-Riemannian manifold. For any $E \in G_2(M,g)$ we define the* **sectional curvature**

$$\sec(E) := \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

where X,Y is a basis of E.

For any $L \in G_1(M,g)$ we define the **Ricci curvature**

$$\mathrm{RIC}(L) := \frac{\mathrm{ric}(X,X)}{g(X,X)},$$

where $L = \mathbb{R}X$.

Both quantities are well-defined, which is obvious for RIC, but this should by proven for sec. We define

$(X,Y)\cdot A = (\tilde{X},\tilde{Y})$

$$\sec(X,Y) := \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

if $\operatorname{span}\{X,Y\} \in G_2(\mathrm{T}_pM, g_p)$. It is immediate to show for $\lambda \neq 0$ we have $\sec(X,Y) = \sec(Y,X) = \sec(\lambda X, Y) = \sec(X, \lambda Y)$.

We claim that $\sec(X,Y) = \sec(X+Y,Y)$. Any matrix $A \in \mathrm{GL}(2,\mathbb{R})$ can be written as a finite composition of the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \; \lambda \in \mathbb{R} \setminus \{0\}.$$

Thus, we can transform any basis of $E = \operatorname{span}\{X,Y\}$ into any other basis by such transformation $(X,Y) \mapsto (Y,X)$, $(X,Y) \mapsto (X+Y,Y)$, $(X,Y) \mapsto (\lambda X, Y)$. Thus the claim implies that $\sec(E)$ does not depend on the choice of basis of E.

To prove the claim, we calculate

$$g(R(X+Y,Y)Y, X+Y) = g(R(X,Y)Y, X+Y) = g(R(X,Y)Y, X \cancel{+Y})$$

$$\begin{aligned} & g(X+Y,X+Y)g(Y,Y) - g(X+Y,Y)^2 \\ = \;& g(X,X)g(Y,Y) + \cancel{2g(X,Y)g(Y,Y)} + \cancel{g(Y,Y)^2} \\ & - g(X,Y)^2 - \cancel{2g(Y,Y)g(X,Y)} - \cancel{g(Y,Y)^2} \\ = \;& g(X,X)g(Y,Y) - g(X,Y)^2 \end{aligned}$$

Remarks 2.1.18 (See also: online exercise sheet for the first week).

(a) If $(e_1, \ldots, e_n)$ is a generalized orthonormal basis of T_pM, $\epsilon_i := g(e_i, e_i) = \pm 1$. Then we have (no Einstein summation!)

$$\mathrm{RIC}(\mathbb{R}e_i) := \epsilon_i \operatorname{ric}(e_i, e_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \underbrace{(\epsilon_i\epsilon_j g(R(e_j, e_i)e_i, e_j))} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \sec\left(\operatorname{span}\{e_i, e_j\}\right).$$

see(span{e_i, e_j})

In this sense Ricci curvature in the direction of X is the "$n-1$ times the average" over the sectional curvatures of "the" planes containing X. In the above formula we define this average by considering the coordinate planes. When the set of all such planes is compact (which happens e.g. if g is Riemannian, or if g is Lorentzian and X timelike), then one can show that $\mathrm{RIC}(\mathbb{R}X)$ is "$n-1$ times the average" of the sectional curvatures over all planes containing X. In particular, the sectional curvature determines the Ricci curvature.

(b) If $X+Y$ and $X-Y$ are not lightlike and non-zero, then, we get $\operatorname{ric}(X,Y)$ by interpolation out of RIC:

ric(X±Y, X±Y)

$$\operatorname{ric}(X,Y) = \frac{1}{4}\Big(\mathrm{RIC}(\mathbb{R}(X+Y))g(X+Y, X+Y) - \mathrm{RIC}(\mathbb{R}(X-Y))g(X-Y, X-Y)\Big).$$

As the space of such (X,Y) is dense in $\mathrm{T}_pM \times \mathrm{T}_pM$, this determines ric.

(c) Sectional determines the Riemann curvature tensor (see exercises).

(d) For $\dim M \geq 4$ Ricci curvature does not determine the Riemann tensor

(e) For $\dim M \geq 3$ scalar curvature does not determine the Ricci tensor

$$
\begin{tikzcd}
R \arrow[r, Rightarrow, shift right] \arrow[d, Leftrightarrow] & \mathrm{ric}/\mathrm{Ric} \arrow[l, Rightarrow, shift right, "\text{only if dim} \leq 3"'] \arrow[d, Leftrightarrow] & \\
\mathrm{sec} \arrow[r, Rightarrow] & \mathrm{RIC} \arrow[r, Rightarrow, shift left] & \mathrm{Scal} \arrow[l, Rightarrow, shift left, "\text{only if dim} \leq 2"]
\end{tikzcd}
$$

(f) For $\dim M = 3$ Ricci curvature determines the Riemann curvature tensor and the sectional curvatures (see exercises)

(g) For $\dim M = 2$, we define $\sec(\mathrm{T}_p M) =: K(p)$ and we have

$$\mathrm{ric}(X, Y) = K(p)\, g(X, Y) \text{for } X, Y \in \mathrm{T}_p M,$$

and $\mathrm{scal}(p) = 2K(p)$. Thus the scalar curvature in p determines the Ricci curvature, the Riemann tensor and the sectional curvature in p. If g is Riemannian, then $K(p)$ is called the **Gauß curvature** of (M, g) in p.

Definition 2.1.19. *Let (M, g) be a semi-Riemannian manifold, and $f : M \to \mathbb{R}$ a $\mathcal{C}^1$-function. Then the* **gradient** $\operatorname{grad} f$ *of f is defined as* $\operatorname{grad} f := (\mathrm{d}f)^\#$, *in other words, it is the unique vector field such that*

$$g(\operatorname{grad} f, Y) = \mathrm{d}f(Y)(= \partial_Y f) \quad \forall Y \in \mathscr{X}(M).$$

$$T_p M \underset{\underset{\flat}{\cong}}{\overset{\#}{\rightleftarrows}} T_p^* M \qquad \alpha_j = g_{ij} X^i$$

$$X \longmapsto g(X, \cdot) = \alpha \qquad X^i = g^{ij} \alpha_j$$

2.2 Second fundamental form and the Gauß formula

∇

In this subsection let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold.

Definition 2.2.1. *A submanifold M of $\overline{M}$ is called a* **semi-Riemannian submanifold** *if for every $p \in M$, the tangent space T_pM is a non-degenerate subspace of $(\mathrm{T}_p\overline{M}, \overline{g}_p)$. Then $g_p := \overline{g}|_{\mathrm{T}_pM \times \mathrm{T}_pM}$, $p \in M$ defines a semi-Riemannian metric on M, called the* **induced semi-Riemannian metric***. Note $g = \iota^*\overline{g}$.* 1st fundamental form = induced semi-R. metric

Recall that T_pM is non-degenerate, if and only if, the **normal bundle**

$\dim (\mathrm{T}_pM)^\perp + \dim \mathrm{T}_pM = \dim \mathrm{T}_p\overline{M}$

$$\mathrm{N}_pM := (\mathrm{T}_p\overset{M}{\cancel{N}})^\perp = \left\{ X \in \mathrm{T}_p\overline{M} \;\middle|\; \forall Y \in \mathrm{T}_pM : \overline{g}(X,Y) = 0 \right\}$$

is a complement of T_pM in $\mathrm{T}_p\overline{M}$.

In the following we also write $\iota : M \to \overline{M}$ for the inclusion. We have

$$\iota^*(\mathrm{T}\overline{M}) = \mathrm{T}\overline{M}|_M = \mathrm{T}M \oplus \mathrm{N}M,$$

$= \{ X \in \mathrm{T}_p\overline{M} \mid p \in M \}$

and the direct sum decomposition is orthogonal with respect to $\overline{g}$.

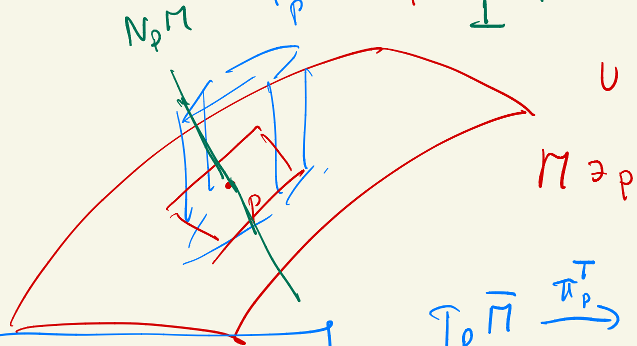
Let $\pi_p^{\mathrm{T}} : \mathrm{T}_p\overline{M} \to \mathrm{T}_pM$ and $\pi_p^{\mathrm{N}} : \mathrm{T}_p\overline{M} \to \mathrm{N}_pM$ be the corresponding orthogonal projections, called the **tangential** and **normal projections**.

Using the inclusion $TM \subset T\overline{M}|_M$ we can view a vector field $Y \in \mathscr{X}(M)$ also as a vector field of $\overline{M}$ along ι, and thus we may derive it in the direction of $X \in \mathscr{X}(M)$ with the Levi-Civita connection of $(\overline{M}, \overline{g})$, which we denote by $\overline{\nabla}$. On the other hand we also may derive Y in the direction of X using the Levi-Civita connection ∇ of (M, g), $g := \iota^*\overline{g}$. Both derivatives are related by the following formula: $Y \in \mathscr{X}(M)$

$X \in \mathscr{X}(M) \rightsquigarrow X \in \Gamma(\iota^* T\overline{M})$ $\quad \overline{\nabla}_Y X \in \Gamma(\iota^* T\overline{M})$

$\rightsquigarrow \nabla_Y X \in \Gamma(TM) = \mathscr{X}(M) \subset \Gamma(\iota^* T\overline{M})$ tangential

$$T_p\bar M = T_pM \overset{\perp}{\oplus} N_pM \qquad \bar M = \mathbb{R}^3$$



$$\mathfrak{X}(\bar M) \ni X : \bar M \to T\bar M$$

$$X|_M : M \to T\bar M$$
$$\in \Gamma(\iota^* T\bar M)$$

$$\begin{array}{ccc} T_p\bar M & \xrightarrow{\pi_p^T} & T_pM \\ \downarrow{\scriptstyle \pi_p^N} & & \\ N_pM & & \end{array}$$

$$\pi^N + \pi^T = id_{T\bar M}$$

$\nabla_X Y - \nabla_Y X = [X, Y]$

Lemma 2.2.2. *For $X, Y \in \mathscr{X}(M)$ we have*

$$\pi^{\mathrm{T}}\left(\overline{\nabla}_X Y\right) = \nabla_X Y, \tag{2.2.1}$$

and $\pi^{\mathrm{N}}\left(\overline{\nabla}_X Y\right) = \pi^{\mathrm{N}}\left(\overline{\nabla}_Y X\right)$.

and

Proof: We want to show Equation (2.2.1) in $p \in M$. There is an open neighborhood U of p in $\overline{M}$ and $\overline{X}, \overline{Y} \in \mathscr{X}(U)$ such that $\overline{X}|_{U \cap M} = X|_{U \cap M}$ and $\overline{Y}|_{U \cap M} = Y|_{U \cap M}$. In other words $\overline{X}$ resp. $\overline{Y}$ is $\iota|_U$-related to $X|_{U \cap M}$ resp. $Y|_{U \cap M}$. Lemma B.2 then implies that $[\overline{X}, \overline{Y}]$ is $\iota|_U$-related to $[X, Y]|_{U \cap M}$. In other words

$U \cap M$ $\quad U \cap M$

$$[\overline{X}, \overline{Y}]|_{U \cap M} = [X, Y]|_{U \cap M}.$$

The analogous result holds for commutators with Z and $\overline{Z}$, if $\overline{Z}$ is $\iota|_U$-related to Z.

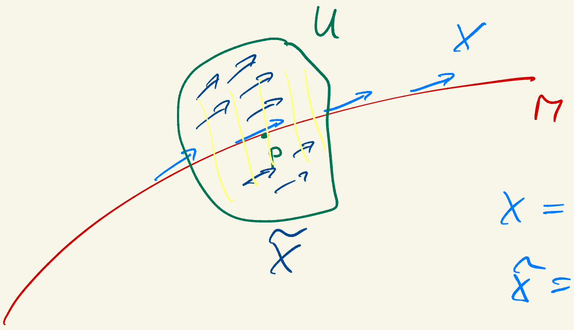
The second statement follows on $U \cap M$ from

$$\begin{aligned} \pi^{\mathrm{N}}\left(\overline{\nabla}_X Y - \overline{\nabla}_Y X\right)|_{U\cap M} \overset{\pi^{\mathrm{N}}\left(\overline{\nabla}_{\overline{X}} \overline{Y} - \overline{\nabla}_{\overline{Y}} \overline{X}\right)|_{U \cap M}}{=} & \ \pi^{\mathrm{N}}\left([\overline{X}, \overline{Y}]|_{U \cap M}\right) \\ = & \ \pi^{\mathrm{N}}\left([X, Y]|_{U \cap M}\right) = 0 \end{aligned}$$

We now use the Koszul formula, see after Theorem 2.1.2, both for $\overline{\nabla}$ and ∇, and we calculate in $p \in M$:

$$\begin{aligned} \overline{g}\Big(\pi^{\mathrm{T}}\left(\overline{\nabla}_X Y\right), Z\Big) &= \overline{g}\big(\overline{\nabla}_X Y, \overline{Z}\big) \\ &= \partial_{\overline{X}} \overline{g}(\overline{Y}, \overline{Z}) + \partial_{\overline{Y}} \overline{g}(\overline{X}, \overline{Z}) - \partial_{\overline{Z}} \overline{g}(\overline{X}, \overline{Y}) \\ &\quad - \overline{g}(\overline{X}, [\overline{Y}, \overline{Z}]) + \overline{g}(\overline{Y}, [\overline{Z}, \overline{X}]) + \overline{g}(\overline{Z}, [\overline{X}, \overline{Y}]) \\ &= \partial_X g(Y, Z) + \partial_Y g(X, Z) - \partial_Z g(X, Y) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \\ &= g\big(\nabla_X Y, Z\big). \end{aligned}$$

Koszul formula



$X = \sum X^i \partial_i$ defined on M

$\tilde X = \sum_{i=1}^{\dim \bar M} X^i \partial_i$ extended

$$X|_{U\cap M} = \bar X|_{U\cap M}$$

$f = \iota|_{U\cap M}$

$$\begin{tikzcd} TM|_{U\cap M} \arrow[r, "df"] \arrow[d, shift left] & T\bar M|_U \arrow[d, shift left] \\ U\cap M \arrow[u, shift left, "X"] \arrow[r, "f"] \arrow[ur, phantom, "\circlearrowright"] & U \overset{\circ}{\subset} \bar M \arrow[u, shift left, "\bar X"'] \end{tikzcd}$$

In fact we could choose $U \supset M$

$$M \xrightarrow{f} N$$

$\bar X \in \mathfrak X(N)$ is f-related to $X \in \mathfrak X(M)$ iff

$$\begin{tikzcd} TM \arrow[r, "df"] \arrow[d, shift right] & TN \arrow[d, shift left] \\ M \arrow[u, shift right, "X"'] \arrow[r, "f"] \arrow[ur, phantom, "\circlearrowright"] & N \arrow[u, shift left, "\bar X"] \end{tikzcd}$$

Lemma

$\bar X$ f-related to X

$\bar Y$ f-related to Y

$\Rightarrow [\bar X, \bar Y]$ f-related to $[X, Y]$

As this holds for all $Z \in \mathscr{X}(M)$, we obtain the first statement. ■

Obviously, for fixed $Y \in \mathscr{X}(M)$, the map

$\mathscr{X}(M) \times \mathscr{X}(M) \to \Gamma(NM)$
$X, Y \mapsto \pi^N(\overline{\nabla}_X Y)$

$$\begin{aligned} \mathscr{X}(M) &\to \Gamma(\mathrm{N}M) \\ X &\mapsto \pi^{\mathrm{N}}(\overline{\nabla}_X Y) \end{aligned}$$

is $\mathcal{C}^\infty(M)$-linear, and because of the symmetry proved in the lemma, $\pi^{\mathrm{N}}(\overline{\nabla}_X Y)$ also depends $\mathcal{C}^\infty(M)$-linearly on Y. Thus there is a section $\vec{\mathrm{II}} \in \Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M \otimes \mathrm{N}M)$, called the **vector-valued second fundamental form** of M in $\overline{M}$, such that

$\vec{\mathrm{II}} \in \Gamma((T^*M \odot T^*M) \otimes NM)$

$$\vec{\mathrm{II}}(X,Y) = \pi^{\mathrm{N}}(\overline{\nabla}_X Y)\,.$$

The preceding lemma implies $\vec{\mathrm{II}}(X,Y) = \vec{\mathrm{II}}(Y,X)$.

In the following we will also allow to write $\langle \bullet, \bullet \rangle$ for g, although we have to keep in mind that $\langle \bullet, \bullet \rangle$ is not necessarily positive definite.

Theorem 2.2.3 (Gauß formula)**.** *Let $M \subset \overline{M}$ be a semi-Riemannian submanifold of $(\overline{M}, \overline{g})$. Then for all $p \in M$ and all $X, Y, Z, W \in \mathrm{T}_pM$* $\subset T_p\overline{M}$ *we have*

$\parallel$
$\langle\, , \,\rangle$

$$\langle R(X,Y)Z,W\rangle = \langle \vec{\mathrm{II}}(X,W), \vec{\mathrm{II}}(Y,Z)\rangle - \langle \vec{\mathrm{II}}(X,Z), \vec{\mathrm{II}}(Y,W)\rangle + \langle \overline{R}(X,Y)Z,W\rangle. \tag{2.2.2}$$

Proof: We calculate for $X, Y, Z, W \in \mathscr{X}(M)$. We use, e. g., $\overline{\nabla}_Y Z = \nabla_Y Z + \vec{\mathrm{II}}(Y,Z)$, and as soon as only the tangential component matters – marked with $(*)$ – , we thus may replace $\overline{\nabla}_Y Z$ by $\nabla_Y Z$.

$$\begin{aligned} \langle \overline{R}(X,Y)Z,W\rangle &= \langle \overline{\nabla}_X \overline{\nabla}_Y Z, W\rangle - \langle \overline{\nabla}_Y \overline{\nabla}_X Z, W\rangle - \langle \overline{\nabla}_{[X,Y]} Z, W\rangle \\ &= \langle \overline{\nabla}_X \big(\nabla_Y Z + \vec{\mathrm{II}}(Y,Z)\big), W\rangle - \langle \overline{\nabla}_Y \big(\nabla_X Z + \vec{\mathrm{II}}(X,Z)\big), W\rangle \\ &\quad - \langle \nabla_{[X,Y]} Z + \vec{\mathrm{II}}([X,Y],Z), W\rangle \end{aligned}$$

$$<\bar{R}(X,Y)Z,W> = -<\vec{\mathrm{II}}(X,W),\vec{\mathrm{II}}(Y,Z)>$$
$$+<\vec{\mathrm{II}}(Y,W),\vec{\mathrm{II}}(X,Z)>$$
$$+<R(X,Y)Z,W>$$

$X,Y,Z,W\in T_pM$

we extend the to vector fields in $\Gamma(\iota^*T\bar{M})$ ^ $\Gamma(TM)^c$

$$\bar{\nabla}_X Y = \nabla_X Y + \vec{\mathrm{II}}(X,Y)$$

$$<\bar{R}(X,Y)Z,W> = <\bar{\nabla}_X\bar{\nabla}_Y Z,W> \text{①}$$
$$+<\bar{\nabla}_Y\bar{\nabla}_X Z,W> \text{②} + <\bar{\nabla}_{[X,Y]}Z,W> \text{③}$$

$$\text{①}\ <\bar{\nabla}_X(\nabla_Y Z+\vec{\mathrm{II}}(Y,Z)),W>$$
$$= <\nabla_X\nabla_Y Z + \cancel{\vec{\mathrm{II}}(X,\nabla_Y Z)},W>$$
$$+<\bar{\nabla}_X\vec{\mathrm{II}}(Y,Z)>$$

$$
\begin{aligned}
&\overset{(*)}{=} \big\langle \overline{\nabla}_X \nabla_Y Z, W \big\rangle + \big\langle \overline{\nabla}_X \vec{\mathrm{II}}(Y,Z)), W \big\rangle \\
&\qquad - \big\langle \overline{\nabla}_Y \nabla_X Z, W \big\rangle - \big\langle \overline{\nabla}_Y \vec{\mathrm{II}}(X,Z)), W \big\rangle \\
&\qquad - \big\langle \nabla_{[X,Y]} Z, W \big\rangle \\
&\overset{(*)}{=} \big\langle \nabla_X \nabla_Y Z, W \big\rangle - \big\langle \nabla_Y \nabla_X Z, W \big\rangle - \big\langle \nabla_{[X,Y]} Z, W \big\rangle \\
&\qquad + \big\langle \overline{\nabla}_X \vec{\mathrm{II}}(Y,Z)\cancel{)}, W \big\rangle - \big\langle \overline{\nabla}_Y \vec{\mathrm{II}}(X,Z)\cancel{)}, W \big\rangle \\
&\overset{(+)}{=} \big\langle R(X,Y)Z, W \big\rangle \\
&\qquad - \big\langle \vec{\mathrm{II}}(Y,Z)\cancel{)}, \vec{\mathrm{II}}(X,W) \big\rangle + \big\langle \vec{\mathrm{II}}(X,Z)\cancel{)}, \vec{\mathrm{II}}(Y,W) \big\rangle .
\end{aligned}
$$

At the last transformation, marked with (+) we *may not* replace $\overline{\nabla}_X$ simply ∇_X as it is not applied to a tangential, but to a normal vector field. We use here the following transformation

$$
\begin{aligned}
\big\langle \overline{\nabla}_X \vec{\mathrm{II}}(Y,Z)), W \big\rangle &= \partial_X \underbrace{\cancel{\langle \vec{\mathrm{II}}(Y,Z)\cancel{)}, W \rangle}}_{=0} - \big\langle \vec{\mathrm{II}}(Y,Z)\cancel{)}, \overline{\nabla}_X W \big\rangle \\
&= -\underbrace{\big\langle \vec{\mathrm{II}}(Y,Z)\cancel{)}, \nabla_X W \big\rangle}_{=0} - \big\langle \vec{\mathrm{II}}(Y,Z\cancel{)}), \vec{\mathrm{II}}(X,W) \big\rangle
\end{aligned}
$$

and the corresponding formula with X and Y replaced. ■

Corollary 2.2.4. *Let M be a semi-Riemannian submanifold of a semi-Riemannian manifold $(\overline{M}, \langle \bullet, \bullet \rangle)$. Let $\overline{\sec}$ be the sectional curvature of $(\overline{M}, \langle \bullet, \bullet \rangle)$ and* sec *the sectional curvature of $(M, \langle \bullet, \bullet \rangle)$, where we also write $\langle \bullet, \bullet \rangle$ for the semi-Riemannian metric on M. Then for any $E \in G_2(M, \langle \bullet, \bullet \rangle) \subset G_2(\overline{M},, \langle \bullet, \bullet \rangle)$ we have*

$$
\sec(E) = \overline{\sec}(E) + \frac{\big\langle \vec{\mathrm{II}}(X,X), \vec{\mathrm{II}}(Y,Y) \big\rangle - \big\langle \vec{\mathrm{II}}(X,Y), \vec{\mathrm{II}}(X,Y) \big\rangle}{\langle X,X \rangle \langle Y,Y \rangle - \langle X,Y \rangle^2}
$$

Example 2.2.5 (Gauß curvature of a surface in $\mathbb{R}^3$)**.** Let M be a surface in Euclidean $\mathbb{R}^3$. At least locally (i.e. on some neighborhood of any given point) there is a unit normal vector field $\nu : M \to S^2$, in other words ν is smooth, $\langle \nu, \nu \rangle \equiv 1$ and $\nu(p) \perp \mathrm{T}_p M$.

Then there is some symmetric $\mathrm{II} \in \Gamma(\mathrm{T}^* M \otimes \mathrm{T}^* M)$ with $\vec{\mathrm{II}}(X, Y) = \mathrm{II}(X, Y)\nu(p)$ for $X, Y \in \mathrm{T}_p M$. There is an orthonormal basis (e_1, e_2) of $\mathrm{T}_p M$ with $\mathrm{II}(e_i, e_j) = \kappa_i \delta_{ij}$, and e_i and e_j are called the **principal curvature directions** of M in p and $\kappa_i \in \mathbb{R}$ the associated **principal curvatures**. Then the Gauß formula says

$$\sec(\mathrm{T}_p M) = \frac{\mathrm{II}(e_1, e_1)\mathrm{II}(e_2, e_2) - \mathrm{II}(e_1, e_2)^2}{1 \cdot 1 - 0} = \kappa_1 \kappa_2,$$

which is the classical Gauß formula.

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2.3 Semi-Riemannian Hypersurfaces

We now specialize to the case that M is **hypersurfaces** in $\overline{M}$, i. e., $\dim M = \dim \overline{M} - 1$. Again M is called a **semi-Riemannian hypersurface** if $g := \overline{g}_{\mathrm{T}M \otimes \mathrm{T}M} = \iota^* \overline{g}$ is non-degenerate. In this case the normal bundle $\mathrm{N}M \to M$ has rank 1. If M is connected this implies that $\overline{g}|_{\mathrm{N}M \otimes \mathrm{N}M}$ is either positive definite (then we say the hypersurface has **sign** $\operatorname{sgn}(M) := +1$, or it is negative definite (then the sign is $\operatorname{sgn}(M) = -1$).

Example 2.3.1. Assume t is a regular value of the smooth function. $f : \overline{M} \to \mathbb{R}$. Then $M := f^{-1}(t)$ is a submanifold, in fact a hypersurface, and $\mathrm{T}_p M = \ker \mathrm{d}_p f = \left(\operatorname{grad} f\big|_p\right)^{\perp}$ for $p \in M$. If $\mathrm{T}_p M$ is a non-degenerate subspace, we can take a generalized orthonormal basis of $\mathrm{T}_p M$ and complete it to a generalized orthonormal basis of $\mathrm{T}_p \overline{M}$ by joining $\sqrt{\left|\langle \operatorname{grad} f\big|_p, \operatorname{grad} f\big|_p \rangle\right|}^{\,-1} \operatorname{grad} f\big|_p$. Thus if $\mathrm{T}_p M$ is non-degenerate, $\operatorname{grad} f$ is non-zero and non-lightlike. Conversely if $\mathrm{T}_p M$ is non-zero and non-lightlike, then $\mathrm{T}_p M = \left(\operatorname{grad} f\big|_p\right)^{\perp}$ is non-degenerate.