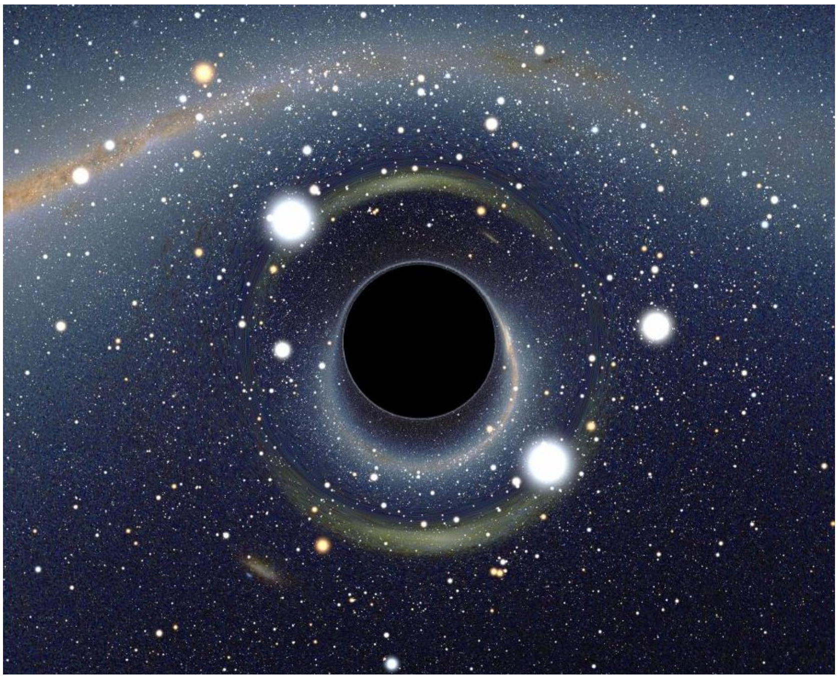
Differential Geometry II
Lorentzian Geometry

Lecture Notes



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Presentation Version

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Wiederholung zu Beginn der Vorlesung am 26.04.:

(V, g) vector space with non-degenerate symmetric form, oriented

Hodge star isomorphism:

$V = \mathbb{R}^{3,1}$

$$*\colon \bigwedge^k V^* \longrightarrow \bigwedge^{n-k} V^*$$

$*(e^1 \wedge e^2) = \pm e^3 \wedge e^4$

$$g(\alpha,\beta)\,\mathrm{vol} = \beta \wedge (*\alpha) \quad \forall \alpha, \beta \in \bigwedge^k V^*.$$

Set $(V, g) := \mathbb{R}^{3,1}$

- electrical field $\vec{E} = (E^1, E^2, E^3)^\top\colon \mathbb{R}\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $(t,x) \mapsto \vec{E}(t,x)$.
- magnetic field $\vec{B} = (B^1, B^2, B^3)^\top\colon \mathbb{R}\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $(t,x) \mapsto \vec{B}(t,x)$.
- the (electrical) charge (density) $\rho\colon \mathbb{R}\times\mathbb{R}^3 \longrightarrow \mathbb{R}$.
- the electrical current (density) $\vec{j}\colon \mathbb{R}\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

$$(F_{\mu\nu})_{\mu\nu} := \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \qquad F := \sum_{0\le\mu<\nu\le 3} F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{1.5.3}$$

in other words $F = \vec{E}^\flat \wedge \mathrm{d}t + *(\vec{B}^\flat \wedge \mathrm{d}t)$ Maxwell equations:

(i) $\mathrm{div}^{\mathbb{R}^3}(\vec{E}) = \rho$. The **Gauß law**.

(ii) $\mathrm{div}^{\mathbb{R}^3}(\vec{B}) = 0$. The **Gauß law for magnetic fields**.

(iii) $\mathrm{rot}(\vec{E}) = -\frac{\partial \vec{B}}{\partial t}$. The **Faraday law**.

(iv) $\mathrm{rot}(\vec{B}) = \vec{j} + \frac{\partial \vec{E}}{\partial t}$. The **Ampère–Maxwell law**.

Ende der Wiederholung

$$dF = \sum_{k=0}^{3} \frac{\partial F_{ij}}{\partial x^k}\, dx^k \wedge dx^i \wedge dx^j$$

$$F = F_{ij}\, dx^i \wedge dx^j$$

We calculate

$$\begin{aligned} \mathrm{d}F &= \mathrm{div}^{\mathbb{R}^3} \vec{B} \cdot \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &\qquad - \underbrace{\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3}_{\mathrm{dvol}} \left(\left(\frac{\partial \vec{B}}{\partial t} + \mathrm{rot}(\vec{E}) \right), \bullet, \bullet, \bullet \right). \end{aligned}$$

As a consequence F is closed if and only if (ii) and (iii) hold.

$$*F = -\vec{B}^\flat \wedge \mathrm{d}t + *(\vec{E}^\flat \wedge \mathrm{d}t)$$

$$\mathrm{dvol} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

$$\begin{aligned} \circledast\; \mathrm{d}(*F) &= (\mathrm{div}^{\mathbb{R}^3} \vec{E}) \cdot \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &\quad + \mathrm{dvol} \left(\left(-\frac{\partial \vec{E}}{\partial t} + \mathrm{rot}(\vec{B}) \right), \bullet, \bullet, \bullet \right) \\ &= \mathrm{dvol} \left(\left((\mathrm{div}^{\mathbb{R}^3} \vec{E}) \frac{\partial}{\partial t} - \frac{\partial \vec{E}}{\partial t} + \mathrm{rot}(\vec{B}) \right), \bullet, \bullet, \bullet \right) \\ &= * \left(\left((\mathrm{div}^{\mathbb{R}^3} \vec{E}) \frac{\partial}{\partial t} - \frac{\partial \vec{E}}{\partial t} + \mathrm{rot}(\vec{B}) \right)^\flat \right). \end{aligned}$$

We define $J(x,t) := \begin{pmatrix} \rho(x,t) \\ \vec{j}(x,t) \end{pmatrix}$, then (i) and (iv) are equivalent to

$$*\mathrm{d}(*F) = J^\flat.$$

Result 1.5.3. *We have written the electromagnetic field as $F \in \Omega^2(V)$, for a $(3+1)$-dimensional Minkowski space. The Maxwell-equations are then equivalent to*

$$\mathrm{d}F = 0, \quad *\mathrm{d}(*F) = J^\flat.$$

Pulling back with any isometry $\mathbb{R}^{3,1} \to V$ and using (1.5.3) we obtain the classical way to write $\vec{E}$ and $\vec{B}$. Any rule how the electromagnetic field changes under change of the inertial system is subsumed into this picture.

Because of the Poincaré lemma there is $A \in \Omega^1(\mathbb{R}^{3,1})$, the **4-vector potential** such that $F = \mathrm{d}A$. This A is unique up to a closed 1-form, as $F = \mathrm{d}A = \mathrm{d}\tilde{A}$ implies $\mathrm{d}(A - \tilde{A}) = 0$, i. e., there is some $h \in \mathcal{C}^\infty(\mathbb{R}^{3,1})$ with $A - \tilde{A} = \mathrm{d}h$.

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As a next step we want to understand the electromagnetic field as connection on a bundle, more precisely as a **complex vector bundle** of rank 1. In [12, Sec. 1.2.1] you have seen the definition of real vector bundles, and by replacing real vector spaces by complex vector spaces (and $\mathbb{R}^k$ by $\mathbb{C}^k$) we obtain the definition of a complex vector bundle. The dimension of the fibers is called the **rank** of the bundle, thus a complex vector bundle of rank k yields a real vector of rank $2k$ if we forget fiberwise multiplication by i, only keeping multiplicatin by real numbers. Real or complex cector bundles of rank 1 are called **line bundles**.

$\boxed{\pi \circ s = \mathrm{id}_M}$

Definition 1.5.4 (Curvature of a vector bundle). *If $V \underset{\pi}{\overset{s}{\rightleftarrows}} M$ is a vector bundle over a manifold M and if ∇^V is a connection on V, then we define the* **curvature** $R^{(V,\nabla^V)}$ *as*

$$R^{(V,\nabla^V)}(X,Y)s = \nabla^V_X \underbrace{\nabla^V_Y s}_{\in \Gamma(V)} - \nabla^V_Y \nabla^V_X s - \nabla^V_{[X,Y]} s, \quad \in \Gamma(V)$$

where $X, Y \in \Gamma(\mathrm{T}M)$ and $s \in \Gamma(V)$. $= \{ s : M \to V \mid \pi \circ s = \mathrm{id}_M \}$

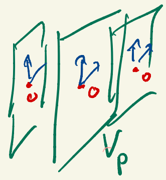
One checks that the map

$$\begin{aligned} R^{(V,\nabla^V)} : \Gamma(\mathrm{T}M) \times \Gamma(\mathrm{T}M) \times \Gamma(V) &\to \Gamma(V) \\ (X, Y, s) &\mapsto R^{(V,\nabla^V)}(X,Y)s \end{aligned}$$

$R^{V,\nabla^V} \in \Gamma(T^*M \otimes T^*M \otimes \underbrace{V^* \otimes V}_{\mathrm{End}(V)})$

is a trilinear map of $\mathcal{C}^\infty(M)$-modules.
This implies that $R^{(V,\nabla)}$ is given by an element of $\Gamma\big(\Lambda^2 \mathrm{T}^*M \otimes \underbrace{\mathrm{End}(V)}_{= \mathbb{C}}\big) =: \Omega^2\big(M, \mathrm{End}(V)\big)$.

$\mathrm{rk}\, V = 1$



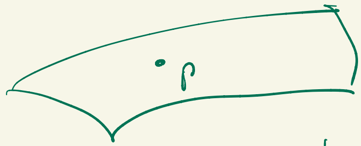
$V_p = \mathbb{C}^k$

$= V \quad \dim_{\mathbb{R}} = n + 2k$

$$\begin{tikzcd} V \arrow[d, "\pi_1"', "\pi^V"] \arrow[r, shift left] & M \times \mathbb{C}^k \arrow[l, shift left] \arrow[d] & (p, v) \arrow[d, mapsto] \\ M & M \times \mathbb{C}^k & (p, H(p)v) \end{tikzcd}$$

$M \quad \dim = n$

$H(p) \in Gl(k, \mathbb{C})$



complex vector bundle

$k = 1$ complex line bundle

$Gl(1, \mathbb{C}) = \mathbb{C} \setminus \{0\}$

$\mathbb{R}^{3,1}$
$\parallel$

Now let us assume that M is a manifold and $A \in \Omega^1(M)$.

$s_f(p) = (p, f(p))$

$$L := M \times \mathbb{C} \xrightarrow{\pi} M$$

$M \times \mathbb{C}^k$

$\pi \circ s = id$

For $p \in M$ the fiber over p is $\pi^{-1}(p) = \{p\} \times \mathbb{C}$, and we use the standard structure as 1-dimensional complex vector space on $\mathbb{C} \cong \{p\} \times \mathbb{C}$. It is thus a complex vector bundle of rank 1, i. e., a complex line bundle. It has a global trivialization, an – up to isomorphism of bundle – there is a unique such bundle (for each rank), there we call L the **trivial complex line bundle** over M. A (smooth) section of L is a (smooth) map $M \to M \times \mathbb{C}$ that is the identity in the first component.
Thus we may identify sections of L with complex-valued functions. Now we define a connection ∇^A on L. For $X \in \Gamma(TM)$ and $s_f \in \Gamma(L)$ we define

$$\left(\nabla^A_X s_f\right)\Big|_p := \Big(p, \partial_{X|_p} f + \overbrace{iA\big(X|_p\big)}^{\in iC^\infty(M)} f\Big)$$

and one can easily check that this satisfies the required properties of a connection claimed in [12, Def. 2.2.1]. The connection also satisfies the product rule

$$\partial_X \langle s_f, s_h\rangle = \overbrace{\langle \nabla^A_X s_f, s_h\rangle + \langle s_f, \nabla^A_X s_h\rangle}^{\in C^\infty(M,\mathbb{C}) \ni},$$

where the fiberwise scalar product is defined by $\langle (p,z),(p,w)\rangle := z\overline{w}$. Connection on complex vector bundles with fiberwise scalar product satisfying this product rule are called **unitary connections**.

We calculate the curvature as

$$\begin{aligned}
\big(R^A(X,Y)s_f\big)\Big|_p &= \Big(\nabla^A_X\nabla^A_Y s_f - \nabla^A_Y\nabla^A_X s_f - \nabla^A_{[X,Y]} s_f\Big)\Big|_p \\
&= \Big(p, \partial_{X|_p}\big(\cancel{\partial_{Y|_p} f} + iA(Y|_p) f\big) + iA(X|_p)\big(\partial_{Y|_p} f + iA(Y|_p)\big) \\
&\qquad -\partial_{Y|_p}\big(\cancel{\partial_{X|_p} f} + iA(X|_p) f\big) - iA(Y|_p)\big(\partial_{X|_p} f + iA(X|_p)\big) \\
&\qquad -\cancel{\partial_{[X,Y]|_p} f} - iA([X,Y]|_p) f\Big)
\end{aligned}$$

$\mathrm{rk}_{\mathbb{C}} V = \cancel{k}\, k \qquad A \in \Omega^1(M, \underbrace{\mathrm{End}(V)}_{\cong \mathbb{C}^{k \times k}})$

$$= \Bigg(p, i\Big(\underbrace{\partial_{X|_p}\big(A(Y|_p)\big) - (\partial_{Y|_p}\big(A(X|_p)\big) - A([X,Y]|_p)}_{\mathrm{d}A(X,Y)}\Big) f$$

$$+ i^2 \Big(\underbrace{A(X|_p)\, A(Y|_p) - A(Y|_p)\, (A(X|_p)}_{=0}\Big) f\Bigg).$$

$$= \; i dA(X,Y)\, s_f$$

Curvature of $L = M \times \mathbb{C}$, ∇^A is $\quad idA = iF$

Now the trivialization of L used above is not unique, other trivializing maps are given by $H \in \mathcal{C}^\infty(M, \mathbb{C} \setminus \{0\})$:

$$T_H : L = M \times \mathbb{C} \to M \times \mathbb{C}, \quad (p, z) \mapsto (p, H(p)z).$$

In this trivialization the form A has to be adapted, due to the calculation $\Gamma(L)$ $\qquad f(p)$

$$\begin{aligned} T_H(\nabla^A_X (T_H)^{-1}(s_f)) &= \Big(p, H(p)\, \partial_{X|_p}(H(\bullet)^{-1} f) + i\cancel{H(p)} A\big(X|_p\big)(\cancel{H(p)}^{-1} f)\Big) \\ &= \Big(p, \partial_{X|_p} f - \frac{\partial_{X|_p}(H)}{H(p)} f + iA\big(X|_p\big) f\Big). \end{aligned}$$

$f(p)$

If M is simply-connected, then there is a function $h \in \mathcal{C}^\infty(M, \mathbb{C})$ with $H = \exp(ih)$, and we get

$$\frac{\partial_{X|_p}(H)}{H(p)} = i\partial_{X|_p} h = i\, \mathrm{d}h(X)|_p,$$

and finally

$$T_H(\nabla^A_X (T_H)^{-1}(s_f)) \;=\; \Big(p, \partial_{X|_p} f + i(A - \mathrm{d}h)\big(X|_p\big) f\Big).$$

If the change of trivialization preserves the fiberwise scalar product, then $H : M \to S^1 \subset \mathbb{C}$ and then then we may choose $h \in \mathcal{C}^\infty(M, \mathbb{R})$.

$\hat{A} = A \pm dh$

What does these bundle theoretic facts provide for electromagnetic fields?

Let $M = \mathbb{R}^{3,1}$, and $F \in \Omega^2(\mathbb{R}^{3,1})$ the 2-form describing the electromagnetic field. Then any 4-potential $A \in \Omega^1(\mathbb{R}^{3,1})$ with $\mathrm{d}A = F$ yields yields a connection ∇^A on the trivial complex line bundle with $R^A(X,Y)s_f = i\,F(X,Y)s_F$.

Result 1.5.5. *We can model the electromagnetic field as a unitary connection on a complex line bundle (with fiberwise scalar product) over $\mathbb{R}^{3,1}$. ~~Two~~ electro magnetic fields are physically undistinguishable if and only if the bundles with connection are isomorphic.*

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Remark 1.5.6. In our argumentation we used that the de Rham cohomology groups $H^k(\mathbb{R}^{3,1})$ vanish for $k = 1, 2$. There are physical situation, namely in quantum mechanics, where it is reasonable to allow more complicated topology for the space, e. g., if we assume the existence of magnetic monopoles or if we do quantum mechanics on completements of wires.

What we did above are only the first steps. Further steps are:

- The above form of the Maxwell equations generalizes immediately to Lorentzian manifolds, i. e., to curved spacetimes.
- It is now easy to view the Maxwell equations as the Euler-Lagrange equation of an action functional in the perspective of Lagrangian mechanics.
- The above considerations lead to describing the electromagnetic field as the curvature of a bundle. This bundle can be enlarged to a larger bundle which then describes the unification of the electromagnetic force with the weak force to the electroweak force. We obtain the "standard model of particle physics".

2 Submanifolds of semi-Riemannian manifolds

2.1 Semi-Riemannian manifolds

In this section we will always assume $n = \dim M$.

Definition 2.1.1. *Let M be a (smooth) manifold. A* **semi-Riemannian metric** *is a (smooth) $(0,2)$-tensor $g \in \Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M)$, such that for all $p \in M$ the associated map $g_p : \mathrm{T}_pM \times \mathrm{T}_pM \to \mathbb{R}$ is a non-degenerate symmetric form. We say that g* **is of signature** *(m,k) if all g_p are of that signature. A* **semi-Riemannian manifold** *(of signature (m,k)) is a pair (M,g) consisting of a smooth manifold M and a semi-Riemannian metric g (of signature (m,k)) on M. An m-dimensional* **Riemannian manifold/metric** *is a semi-Riemannian manifold/metric of signature $(m,0)$, and an $(m+1)$-dimensional* **Lorentzian manifold/metric** *is a semi-Riemannian manifold/metric of signature $(m,1)$.*

$\mathfrak{X}(M) = \Gamma(TM)$

Theorem 2.1.2. *Let (M, g) be a semi-Riemannian manifold. then there is a unique map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ with the properties: For $f \in \mathcal{C}^\infty(M)$, $X, Y, Z \in \mathfrak{X}(M)$, $\alpha \in \mathbb{R}$ we have:*

affine

(i) $\nabla_{fX} Y = f \nabla_X Y$ *und* $\nabla_{X+X_1} Y = \nabla_X Y + \nabla_{X_1} Y$. (**$\mathcal{C}^\infty(M)$-linear in the 1st argument**)
In other words $X \mapsto \nabla_X Y$ is a $\mathcal{C}^\infty(M)$-modul homomorphismus.

(ii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$. (**additive in the 2. argument**)
Together with $\nabla_X(\alpha Y) = \alpha \nabla_X Y$ we obtain $\mathbb{R}$-linearity in the 2. argument, i. e., the map $Y \mapsto \nabla_X Y$ is an $\mathbb{R}$-linear map.

(iii) $\nabla_X (fY) = (\partial_X f) Y + f \nabla_X Y$. (**Product rule I**)

(iv) $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ (**torsion free**)

(v) $\partial_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$. (**Product rule II**)

Deriving with respect to the Levi-Civita connection is also called **covariant derivation**.

The proof is very similar as in the Riemannian case, so we do not want to give it in details, see [13, Chap. 3] for a full proof. Let us mention instead what has to be modified in the proof of Analysis IV, Kapitel 2, Satz 2.7.2 [4] to get a proof of the above theorem. In the proof of uniqueness one assumes the existence of two connections ∇ and $\widetilde{\nabla}$ with the required properties. For any $X, Y, Z \in \Gamma(\mathrm{T}M)$ one can show with identical arguments that

$g(\nabla_X Y - \widetilde{\nabla}_X Y, Z) = 0$

$$g(\nabla_X Y, Z) - g(\widetilde{\nabla}_X Y, Z) = 0. \qquad (2.1.1)$$

The goal is to show that $\nabla_X Y = \widetilde{\nabla}_X Y$. In the Riemannian case one can simply apply (2.1.1) to $Z := \nabla_X Y - \widetilde{\nabla}_X Y$, and we get $g(Z, Z) = 0$ and so $Z = 0$. This argument is no longer valid, but obviously the non-degeneracy of g yields the conclusion as well.

Lemma 2.1.3 (Locality of the Levi-Civita connection)**.** *Assume $V \Subset U \Subset M$, let ∇^M and ∇^U be the Levi-Civita connections of (M,g) and $(U, g|_U)$. Let $X, Y \in \Gamma(\mathrm{T}M)$ and $\widetilde{X}, \widetilde{Y} \in \Gamma(\mathrm{T}U)$ with $X|_V = \widetilde{X}|_V$ and $Y|_V = \widetilde{Y}|_V$, then $\left(\nabla^M_X Y\right)|_V = \left(\nabla^U_{\widetilde{X}} \widetilde{Y}\right)|_V$.*

Notation 2.1.4. Let $x : U \to V$, $U \overset{\circ}{\subset} M$, $V \overset{\circ}{\subset} \mathbb{R}^n$ be a chart of the semi-Riemannian manifold (M,g) with coordinate vector fields $(\partial_1, \ldots, \partial_n)$. We define $g_{ij} := g(\partial_i, \partial_j) \in \mathcal{C}^\infty(U)$. We assume that $(g^{ij}(p))_{ij}$ is the matrix inverse to $(g_{ij}(p))_{ij}$, so also $g^{ij} \in \mathcal{C}^\infty(U)$. $\partial_i = \frac{\partial}{\partial x^i}$

$\overset{\circ}{\subset}$, $V \overset{\circ}{\subset} \mathbb{R}^n$

Definition 2.1.5 (Christoffel symbols)**.** *Let $x : U \to V$, $U \overset{\circ}{\subset} M$, be a chart of the semi-Riemannian manifold (M,g) and we use Notation 2.1.4. Then we define the* **Christoffel symbols** $\Gamma^k_{ij} : U \to \mathbb{R}$ *of (M,g) for the chart x by the formula*

$$\sum_{k=1}^{n} \Gamma^k_{ij} \partial_k := \nabla_{\partial_i} \partial_j.$$

Here we use the convention, that we always sum over summands which contain an index both as an upper and as a lower index. This is called the **Einstein notation**, *also called the* **Einstein summation convention**.

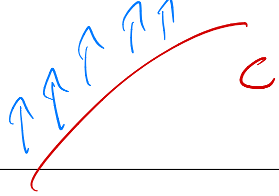
The properties in Theorem 2.1.2 imply

$$\Gamma^k_{ij} = \frac{1}{2} \left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij} \right) g^{km}.$$

As a consequence we have for $X = X^i \partial_i$ and $Y = Y^j \partial_j$:

$$\nabla_X Y = X^i \left((\partial_i Y^k) + Y^j \Gamma^k_{ij} \right) \partial_k.$$

This formula implies that $(\nabla_X Y)|_p$ only depends on on $X|_p$, $Y|_p$ and the derivatives of the components of Y, expressed in some coordinates,



in the direction of X. In particular, if $c : (-\epsilon, \epsilon) \to M$ is a curve with $\dot{c}(0) = X$, and if Y is a vector field along the curve c, then $\nabla_X Y$ is well-defined. This derivative will be denoted by $\frac{\nabla}{\mathrm{d}t} Y$. Let us formalize this a bit clearer.

Definition 2.1.6. *Let N and M be smooth manifolds, and $f \in \mathcal{C}^\infty(N, M)$. Let $\pi^{\mathrm{T}M} : \mathrm{T}M \to M$ be the standard footpoint projection. A* **vector field along** f *is a smooth map $X : N \to \mathrm{T}M$ such that $\pi^{\mathrm{T}M} \circ X = f$*

$$\begin{tikzcd} f^*(TM) \arrow[r] \arrow[d] & \mathrm{T}M \arrow[d, "\pi^{\mathrm{T}M}"] \\ N \arrow[ur, dashed, "X"] \arrow[r, "f"] & M \\ f^{-1}(U) \arrow[u, phantom, "\cup"] \arrow[r] & U \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$f^*(TM) = \{(p, v) \in N \times TM \mid f(p) = \pi^{TM}(v)\}$

pullback

The space of all vector fields along f will be called $\Gamma(f^*TM)$.

If $x : U \to V$, $U \Subset M$, $V \Subset \mathbb{R}^n$ is a chart of M, then every vector field along x can be written as

$$X\big|_p = X^i(p)\partial_i\big|_{f(p)},$$

where $X^i : f^{-1}(U) \to \mathbb{R}$ are smooth.

Now, let us assume that M carries a semi-Riemannian manifold, and let ∇ be the associated Levi-Civita connection. For X as above and $Y \in \mathrm{T}N$ we define

$$\nabla_Y X = (\partial_Y X^i)\partial_i + X^i \left(\nabla_{\mathrm{d}f(Y)} \partial_i\right) \circ f. \tag{2.1.2}$$

Examples 2.1.7.

(1) In the case $N = M$, $f = \mathrm{id}_M$ this is just the Levi-Civita connection defined before.

(2) In the case that f is constant, say $f \equiv p \in M$, this is just the usual derivative $\nabla_Y X = \partial_Y X$ of $X : N \to \mathrm{T}_p M$.

(3) If N is a submanifold of M, and $f = \iota : N \hookrightarrow M$ the inclusion, this defines a connection on the bundle

$TM = \dot{\bigcup}_{p \in M} T_p M$

$$TM\big|_N := \iota^* TM = \dot{\bigcup_{p \in N}} \mathrm{T}_p M$$

$TN = \dot{\bigcup}_{p \in N} T_p N$

viewed as a vector bundle over N.

(4) If I is an interval and $\gamma : I \to M$ a smooth curve. Then $\dot{\gamma}(t) = (d/dt)\,\gamma(t)$ is a vector field along γ. Thus $\dot{\gamma} \in \Gamma(\gamma^* \mathrm{T}M)$.

Every $Y \in \Gamma(\gamma^* \mathrm{T}M)$ can be derived with respect to t, using ∇. For this derivative we write

$$\frac{\nabla}{\mathrm{d}t} Y := \nabla_{\frac{\partial}{\partial t}} Y.$$

In particular, we have defined the second derivative

$$\frac{\nabla}{\mathrm{d}t} \dot{\gamma}(t)$$

of a curve γ.

Definition 2.1.8. *Let (M, g) be a semi-Riemannian manifold. A curve $\gamma : I \to M$ is a* **geodesic**, *if, and only if for all $t \in I$:*

$$\frac{\nabla}{\mathrm{d}t} \dot{\gamma}(t) = 0.$$

One easily checks that γ is a geodesic if and only if for any coordinate system $x : U \to V$, we have – using Notation 2.1.4 and $x \circ \gamma = (\gamma^1, \dots, \gamma^n)^\intercal$: for all $t \in \gamma^{-1}(U)$ we have

$$\ddot{\gamma}^k(t) = -\Gamma_{ij}{}^k \dot{\gamma}^i(t)\dot{\gamma}^j(t).$$

Lemma 2.1.9. *If $\gamma : I \to M$ is a geodesic, then $t \mapsto g(\cancel{\dot{c}}(t), \cancel{\dot{c}}(t))$ is constant.*

γ γ

Proof:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} g\big(\dot{\gamma}(t), \dot{\gamma}(t)\big) &= \partial_{\dot{\gamma}(t)} g\big(\dot{\gamma}(t), \dot{\gamma}(t)\big) \\
&= g\big(\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t), \dot{\gamma}(t)\big) + g\big(\dot{\gamma}(t), \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)\big) \\
&= 2g\Big(\underbrace{\frac{\nabla}{\mathrm{d}t}\dot{\gamma}(t)}_{=0}, \dot{\gamma}(t)\Big) = 0
\end{aligned}$$

■

For vectors and ~~curves~~ we define the notations of **timelike**, **spacelike**, **lightlike**, **nonspacelike** and ~~**causal**~~ as in Definitions 1.2.2 and 1.2.6.

Corollary 2.1.10. *For a* non-constant *geodesic γ in a semi-Riemannian manifold exactly one of the following is true*

(1) *γ is timelike, i. e., for all t we have $g(\dot{\gamma}(t), \dot{\gamma}(t)) < 0$;*

(2) *γ is lightlike, i. e., for all t we have $g(\dot{\gamma}(t), \dot{\gamma}(t)) = 0$;*

(3) *γ is spacelike, i. e., for all t we have $g(\dot{\gamma}(t), \dot{\gamma}(t)) > 0$.*

Note that geodesics may be used to define **normal coodinates** with similar definitions and properties as in the Riemannian situation.

Definition 2.1.11. *On a semi-Riemannian manifold (M, g) we define the following curvature tensor.*

(a) *For vector fields $X, Y, Z \in \Gamma(\mathrm{T}M)$ we define the* **Riemann curvature tensor**

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \in \Gamma(\mathrm{T}M).$$

One proves as in the Riemannian context that R is a $(1,3)$-tensor (field).

(b) *We define the* **Ricci curvature tensor** *or the* **Ricci curvature $(0,2)$-tensor** $\mathrm{ric} \in \Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M)$ *as*

$$\forall p \in M : \forall X, Y \in \mathrm{T}_pM : \mathrm{ric}(X,Y) := \mathrm{tr}\big(W \mapsto R(W,X)Y\big).$$

(c) *The* **Ricci curvature endomorphism** $\mathrm{Ric} \in \Gamma(\mathrm{T}M \otimes \mathrm{T}^*M)$ *as the unique tensor such that for all $p \in M$ and all $X, Y \in \mathrm{T}_pM$:*

$$g(\mathrm{Ric}(X), Y) = \mathrm{ric}(X,Y).$$

(d) *the* **scalar curvature**

$$\mathrm{scal} := \mathrm{tr}\,\mathrm{Ric} \in \mathcal{C}^\infty(M).$$

a smooth map

A map $f : M \to N$ manifolds is called a **local diffeomorphism** if for every $p \in M$ there are $U \Subset M$, $V \Subset N$ with $p \in U$ and $f(p) \in V$ such that $f|_U : U \to V$ is a diffeomorphism. If g (resp. $\hat{g}$) is a semi-Riemannian metric on M (resp. N), then we say that this $f : M \to N$ is a **local isometry** from (M, g) to $(N, \hat{g})$ if for all $p \in M$ the differential $\mathrm{d}_pf : \mathrm{T}_pM \to \mathrm{T}_{f(p)}N$ is an isometry. Obviously any local isometry is a local diffeomorphism. If a local isometry f is also a bijective, then f is called an **isometry**.

Lemma 2.1.12. *If* (M,g) *and* (N,h) *are semi-Riemannian manifolds and if* $f : M \to N$ *is a local isometry, then for all* $p \in M$ *and all* $X,Y,Z,W \in \mathrm{T}_pM$*:*

(a) $\mathrm{d}_pf\left(R^{(M,g)}(X,Y),Z\right) = R^{(N,h)}\big(\mathrm{d}_pf(X),\mathrm{d}_pf(Y)\big)\mathrm{d}_pf(Z)$.

(b) $f^*\,\mathrm{ric}^{(N,h)} = \mathrm{ric}^{(M,g)}$.

(c) $\mathrm{ric}^{(N,h)}\big(\mathrm{d}_pf(X)\big) = \mathrm{d}_pf\big(\mathrm{Ric}^{(M,g)}(X)\big)$

(d) $\mathrm{scal}^{(M,g)} = \mathrm{scal}^{(N,h)} \circ f$.

■

Proposition 2.1.13 (Symmetries of the curvature tensors)**.** *Let* (M,g) *be a semi-Riemannian manifold. Then for all* $p \in M$ *and all* $X,Y,Z,W \in \mathrm{T}_pM$ *we have*

(1) $g(R(X,Y)Z,W) = -g(R(Y,X)Z,W)$ (**Anti-symmetry in first two slots**)

(2) $g(R(X,Y)Z,W) = -g(R(X,Y)W,Z)$ (**Anti-symmetry in last two slots**)

(3) $g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$ (**Interchange symmetry**)

(4) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (**1st Bianchi identity**)

The proposition is a generalization of [12, Prop. 2.4.9], and can be proved completely analogously. ■

Note that the last inequality, i. e., the 1st Bianchi identity is also called the **algebraic Bianchi identity**. However, it was not found by Bianchi, but by Ricci. Due to Bianci is the “second Bianchi identity”, and due to their similarity both come under the name Bianchi identity.

Definition 2.1.14 (covariant derivation of tensors)**.** *There are unique connections ∇ on*

$$\mathrm{T}^{(r,s)}M := \underbrace{\mathrm{T}M \otimes \cdots \otimes \mathrm{T}M}_{r\ \textit{times}} \otimes \underbrace{\mathrm{T}^*M \otimes \cdots \otimes \mathrm{T}^*M}_{r\ \textit{times}}$$

such that

(1) *on* $\mathcal{C}^\infty(M) = \Gamma(\mathrm{T}^{(0,0)}M)$ *we have* $\nabla = \partial$,

(2) *on* $\Gamma(\mathrm{T}M)$ *the connection* ∇ *is the Levi-Civita connection*

(3) *for* $\alpha \overset{\in}{} \Gamma(\mathrm{T}^*M)$ *we have*

$\alpha \in \Omega^1(M($

$$\textit{for all } X, Y \in \Gamma : \ (\nabla_X \alpha)(Y) = \partial_X(\alpha(Y)) - \alpha(\nabla_X Y).$$

(4) *for* $\cancel{t} \in \Gamma(\mathrm{T}^{(r,s)})$ *and* $\cancel{\hat{t}} \in \Gamma(\mathrm{T}^{(\hat{r},\hat{s})})$ *we have*

t_1 t_2

$$\nabla_X(t_1 \otimes t_2) = (\nabla_X t_1) \otimes t_2) + t_1 \otimes \nabla_X t_2)$$

Theorem 2.1.15 (2nd Bianchi identity/differential Bianchi identity). *Then for all $p \in M$ and all $X, Y, Z, W \in \mathrm{T}_p M$ we have*

$$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0.$$

This identity is called the **2nd Bianchi identity** or the **differential Bianchi identity**. **Proof:** It is sufficient to proof the theorem if X, Y, Z are coordinate vector fields for normal coordinates centered in $p \in M$. We then have

$$\begin{array}{ccc} \nabla_X X\big|_p = 0 & \nabla_X Y\big|_p = 0 & \nabla_X Z\big|_p = 0 \\ \nabla_Y X\big|_p = 0 & \nabla_Y Y\big|_p = 0 & \nabla_Y Z\big|_p = 0 \\ \nabla_Z X\big|_p = 0 & \nabla_Z Y\big|_p = 0 & \nabla_Z Z\big|_p = 0 \end{array}$$

and thus also all commutators of X, Y, and Z vanish at p.

$$\begin{aligned} (\nabla_X R)(Y,Z)W\big|_p &= \nabla_X (R(Y,Z)W)\big|_p - \underbrace{R(\nabla_X Y, Z)W\big|_p}_{=0} \\ &\quad - \underbrace{R(Y, \nabla_X Z)W\big|_p}_{=0} - R(Y,Z)\nabla_X W\big|_p \end{aligned}$$

We obtain at the point p:

$$(\nabla_X R)(Y,Z) = [\nabla_X, R(Y,Z)] = \big[\nabla_X, [\nabla_Y, \nabla_Z]\big].$$

The statement now follows from the Jacobi identity which says that for linear maps $A, B, C \in \mathrm{End}(V)$ of a vector space V one has $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$. ■