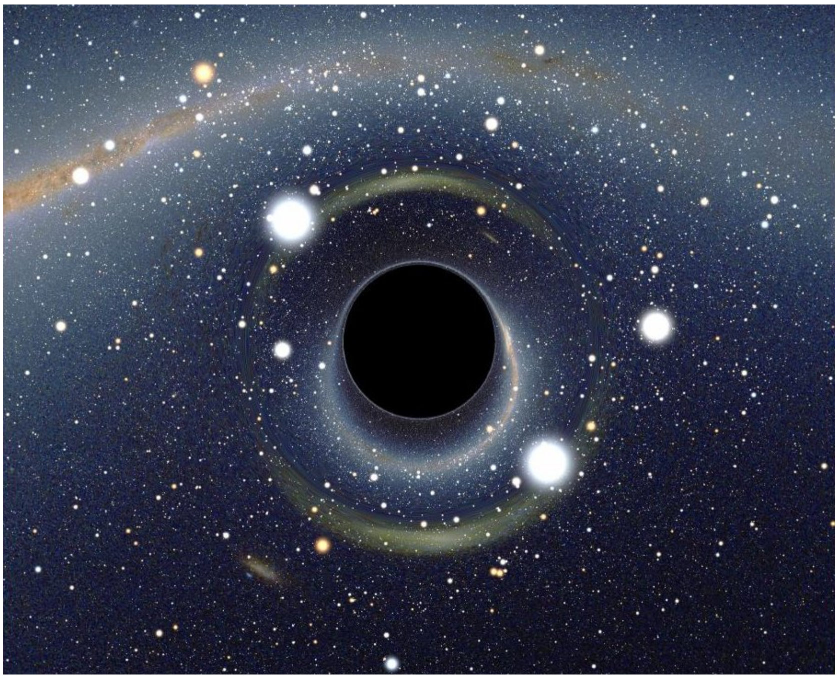
Differential Geometry II
Lorentzian Geometry
Lecture Notes



Prof. Dr. Bernd Ammann

Summer term 2021

University of Regensburg

Presentation Version

Version of April 20, 2021

1.4 Special relativity: Main ingredients

$\mathbb{R}^3 \quad = \mathbb{R} \times \mathbb{R}^3$

In special relativity we combine classical objects to relativistic objects, which are called “4-vectors” by physicists, for example:

classical scalar	name	classical vector	name	relativistic object	name
t	time	$\vec{x}$	position	$x = (t, \vec{x})$	event
E	energy	$\vec{p}$	momentum	$p = (E, \vec{p})$	(energy-)momentum 4-vector
ρ	electrical charge	$\vec{j}$	electrical current	$J = (\rho, \vec{j})$	electrical 4-current
ϕ	electric potential	$\vec{A}$	magnetic potential	$A = (\phi, \vec{A})$	electromagnetic potential

As an example we claim that all freely falling objects move along straight affine lines in $\mathbb{R}^{3,1}$, i. e., along worldlines

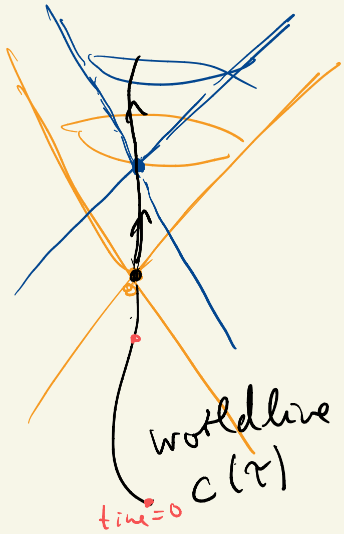
non accelerated

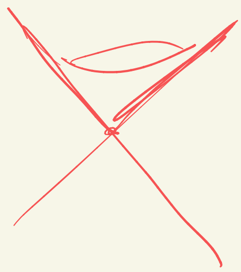
$$\mathbb{R} \ni \tau \in x(\tau) = x_0 + \tau v_0 \in \mathbb{R}^{3,1}$$

where $x_0 \in \mathbb{R}^{3,1}$ and v_0 is causal future-directed, and if the objects has non-zero rest mass m_0 we have the additional requirement that v_0 has to be timelike.

An arbitrary object is decribed by a worldline $\tau \mapsto x(\tau)$, a curve in spacetime. One additionally requires that the derivative $\dot{x}(\tau)$ of $x(\tau)$ is causal future-directed for any τ, and in case of non-zero rest mass it is even requested to be timelike.

Note that if we replace $x(\tau)$ by a reparametrization $y(\tau) := x(\varphi(\tau))$, then $\dot{y}(\tau) = \dot{\varphi}(\tau)\dot{x}(\varphi(\tau))$ and thus the conditions above on $\dot{x}(\tau)$ do not depend on the choice of parametrization.





Plus

$$\underbrace{\left(\dot{c}(\tau)\right)^0}_{\text{invariant under } O_1(m,1)} > 0$$

$$\dot{c}(\tau) = \left(\frac{d}{d\tau} c\right)$$

massive $m_0 > 0$ $\quad \langle\langle \dot{c}(\tau), \dot{c}(\tau) \rangle\rangle < 0$

$m_0 = 0$ $\quad \langle\langle \dot{c}(\tau), \dot{c}(\tau) \rangle\rangle \leq 0$

Wed 21.4.

The length of a causal curve c will be interpreted as the **proper time**, and it is interpreted as the time felt by an observer along this worldline. It is possible to reparametrize a worldline by proper time, which means that $\langle\langle \dot{c}(\tau), \dot{c}(\tau) \rangle\rangle \equiv -1$. Now one defines the **4-velocity** $v_c(\tau)$ as the derivation of $c(\tau)$ with respect to proper time, thus in an arbitrary parametrization we get

$v_{c\circ\varphi}(\tau) = v_c(\varphi(\tau))$

$$v_c(\tau) = \frac{\dot{c}(\tau)}{\sqrt{-\langle\langle \dot{c}(\tau), \dot{c}(\tau) \rangle\rangle}}.$$

The acceleration an observer along the worldline c will feal is the so-called **4-acceleration**

$$a_c(\tau) := \frac{1}{\sqrt{-\langle\langle \dot{c}(\tau), \dot{c}(\tau) \rangle\rangle}} \frac{d}{d\tau} v_c(\tau) = \frac{1}{\sqrt{-\langle\langle \dot{c}(\tau), \dot{c}(\tau) \rangle\rangle}} \frac{d}{d\tau} \left(\frac{\dot{c}(\tau)}{\sqrt{-\langle\langle \dot{c}(\tau), \dot{c}(\tau) \rangle\rangle}} \right)$$

which is the second derivative of c with respect to proper time..

Note that the **classical velocity** is defined as $\vec{v}_c^{\,\mathrm{cla}}(\tau) := \left(\dot{c}^0(\tau)\right)^{-1} \dot{\vec{c}}(\tau)$ and thus $\dot{c}(\tau)$ is proportional to $\begin{pmatrix} 1 \\ \vec{v}_c^{\,\mathrm{cla}}(\tau)) \end{pmatrix}$. We obtain

$$v_c(\tau) = \frac{1}{\sqrt{1 - \|\vec{v}_c^{\,\mathrm{cla}}(\tau)\|^2}} \begin{pmatrix} 1 \\ \vec{v}_c^{\,\mathrm{cla}}(\tau) \end{pmatrix}.$$

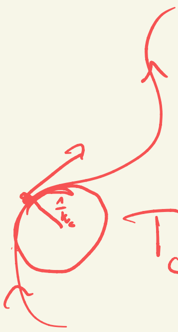
If $c:[a,b] \to \mathbb{R}^{m,1}$ C^1, s.that

$\dot c(t)$ time-like

$$\varphi(t) := \mathcal{L}(c|_{[a,t]}) = \underbrace{\int_a^t \sqrt{-\langle\langle \dot c(\tau), \dot c(\tau)\rangle\rangle}\, d\tau}_{\text{proper time of } c}$$

$$x(\sigma) = c(\varphi^{-1}(\sigma))$$

$$\boxed{\langle\langle \dot x(\sigma), \dot x(\sigma)\rangle\rangle = -1} \overset{\text{def}}{\Longleftrightarrow}: \text{ } x \text{ parametr. by proper time}$$

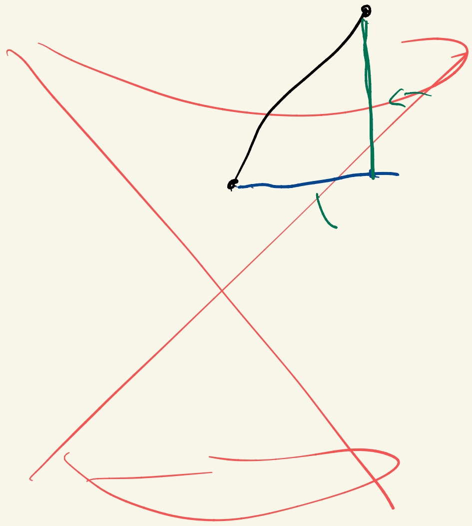


$c : (a,b) \to \mathbb{R}^n$ regular

$$T_c(t) = \frac{\dot c(t)}{\|\dot c(t)\|} \qquad T_{c\circ\varphi}(t) = T_c(\varphi(t))$$

$$k_c(t) = \frac{1}{\|\dot c(t)\|} \frac{d T_c(t)}{dt}$$ derivative wrt arc length

curvature



$$c(\tau) = \begin{pmatrix} c^0(\tau) \\ \vec{c}(\tau) \end{pmatrix}$$

$$\vec{V}_c^{cla}(\tau) = \frac{\frac{d\vec{c}(\tau)}{d\tau}}{\frac{d}{d\tau}c^0(\tau)}$$

$$\dot{c}(\tau) \sim \left(1, \vec{v}_c^{cla}(\tau)\right)$$

$$\Rightarrow v_c(\tau) = \frac{\left(1, v_c^{cla}(\tau)\right)}{\sqrt{1 - \|\vec{v}_c^{cla}(\tau)\|^2}}$$

Example 1.4.1 (Twin paradox)**.** Two twin sisters are born on earth, which we consider as being non-accelerated (which is compared to the speed of light a reasonable approximation). The twins have the role of observers. Twin no. 1 will remain on Earth. At the age of 30 years twin no. 2 is becoming astronaut, and she will travel with 60% the speed of light in the direction of some other star, but at the age of 40 years she decides to return to the earth, flying again with 60% of the speed of light, where she arrives at the age of 50 years. What is the age of her twin sister?

Let $x = (t = x^0, x^1, x^2, x^3)^\top$ be the coordinates measured by twin no. 1, normalized to $t = 0$ at the age of 30 years. She will note that – after her departure – her sister will travel along the worldline

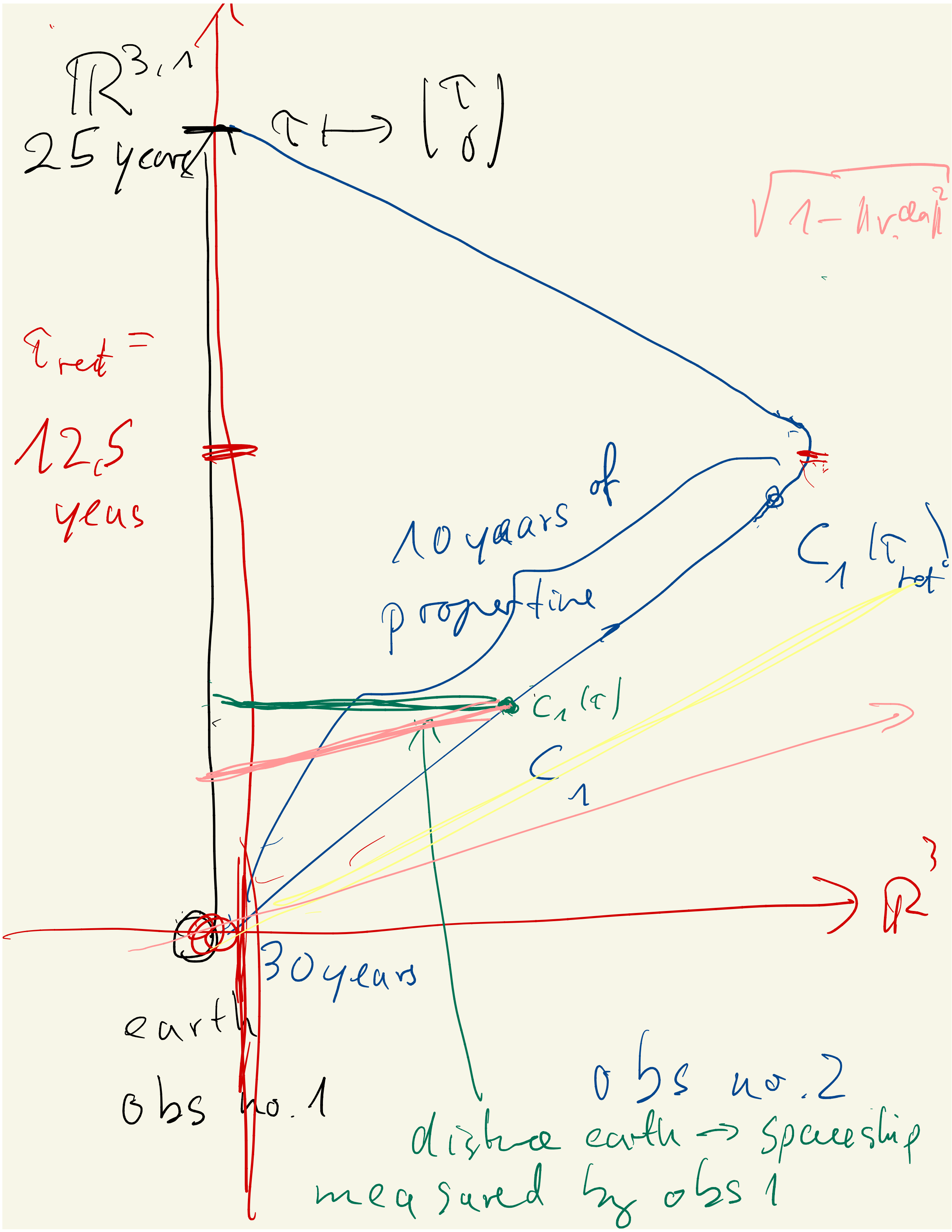
$$c_1 : [0, \tau_{\text{ret}}] \to \mathbb{R}^{3,1}, \quad \tau \mapsto (\tau, 0.6 \cdot \tau, 0, 0)^\tau,$$

until she will reach her return point for $\tau = \tau_{\text{ret}}$. For $0 \le \tau \le \tau_{\text{ret}}$ we calculate $\dot{c}_1(\tau) = (1, 0.6, 0, 0)^\top$ and thus $\langle\!\langle \dot{c}_1(\tau), \dot{c}_1(\tau) \rangle\!\rangle = -0.64$. Thus the proper time of the worldline c_1 is τ_{ret} 12,5

$$10\,y \rightsquigarrow \mathcal{L}(c_1) = \int_0^{\tau_{\text{ret}}} \sqrt{0.64}\, \mathrm{d}\tau = 0.8\tau_{\text{ret}}$$

and as this should be 10 years, we have $\tau_{\text{ret}} = 12.5$ years. So twin no. 1 will say that her sister reached the return point, when she was 42.5 years old.

A similar calculation will show that the trip back also took 12.5 years – from the point of view of sister no. 1. So twin no. 1 will be 5 years older after the trip than sister no. 2.



$\mathbb{R}^{3,1}$
25 years
$\tau \mapsto \begin{pmatrix} \tau \\ 0 \end{pmatrix}$
$\sqrt{1 - \|v_{rel}\|^2}$
$\tau_{ret} =$
12,5 years
10 years of proper time
$C_1(\tau_{ret})$
$C_1(\tau)$
C_1
$\mathbb{R}^3$
30 years
earth
obs no. 1
obs no. 2
distance earth → spaceship measured by obs 1

$\tau \longmapsto \begin{pmatrix} \tau \\ 0{,}6\tau \\ 0 \\ 0 \end{pmatrix} = c_1(\tau)$

Example 1.4.2 (Lorentz contraction)**.** Let us also discuss in the above examples the question, what the distance between the sisters will be, the so-called **Lorentz contraction** or **Lorentz-Fitzgerald contraction**. This distance will also depend on the perspective. From the point of view of sister no. 1, for $0 \le \tau \le \tau_{\text{ret}}$ the distance will be 0.6τ, and for $\tau_{\text{ret}} \le \tau \le 2\tau_{\text{ret}}$, it will be $0.6 \times (2\tau_{\text{ret}} - \tau)$. In particular, the distance point of the return point is $0.6 \times 12.5 = 7.5$ light years.

However, while sister no. 2 is moving, she will experience another distance. For $0 \le \tau \le \tau_{\text{ret}}$ her coordinates $(\hat{t} = \hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3)^\top$ are obtained from her sister's coordinates by the Lorentz boost

$$\begin{pmatrix} \hat{x}^0 \\ \hat{x}^1 \\ \hat{x}^2 \\ \hat{x}^3 \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix},$$

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cosh(-\alpha) & \sinh(-\alpha) & 0 & 0 \\ \sinh(-\alpha) & \cosh(-\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x}^0 \\ \hat{x}^1 \\ \hat{x}^2 \\ \hat{x}^3 \end{pmatrix},$$

and the factor α is determined by setting $\hat{x}^1 = 0$, $x^0 = \tau$, $x^1 = .6\tau$. This leads to $0 = \sinh\alpha + 0.6\cosh\alpha$, i. e., $\tanh\alpha = -0.6$.

Sister no. 2 will see the earth moving away from her along the worldline $\sigma \mapsto (\sigma\cosh\alpha, \sigma\sinh\alpha, 0, 0)^\top$ which after reparametrization is $\hat{\sigma} \mapsto (\hat{\sigma}, -0.6\hat{\sigma}, 0, 0)^\top$, so she will experience the same relative speed, but her time passes slower. Thus, to that spacetimes point $c_1(\tau)$ sister no. 1 measures a distance 0.6τ, while sister no. 2 measures a distance $0.8 \times 0.6\tau = .48\tau$. In particular, shortly before she starts the slowing down process for the return, sister no. 2 will realize a distance of 6 light years, but during the breaking process the distance, detected by sister no. 2 will increase to 7.5 light years, again reducing to 6 light years when she accelerates in the opposite direction.

Let us also add some comments about the energy of a particle moving along a worldline $\tau \mapsto c(\tau)$. Each particle, has a **rest mass** $m_0 \geq 0$ which remains constant along the worldline and which is supposed to take over the classical property that every object has its mass. So assume $m_0 > 0$. In classical mechanics the (kinetic) energy of the particle is said to be given by $E_{\text{kin}}^{\text{class}} = \frac{1}{2} m_0 \|\vec{v}_c^{\text{cla}}\|^2$, but physics would not change if we add a constant E_0, potentially depending on the particle, so we obtain the ansatz

$$E^{\text{class}} = E_0 + E_{\text{kin}}^{\text{class}} = E_0 + \frac{1}{2} m_0 \|\vec{v}_c^{\text{cla}}\|^2.$$

$E = \frac{1}{\sqrt{1-\|\vec{v}^{cl}\|^2}} m_0$

$\cong m_0 + \frac{1}{2} m_0 \|\vec{v}_c\|^2 + O(\|\vec{v}_c\|^4)$

We have to find a relativistic generalization, and it turns out – see [6, Sec. 1.5] for details – that the **relativistic energy** $E := \sqrt{1 - \|\vec{v}_c^{\text{cla}}\|^2}^{\,-1} m_0$ is suitable. Together with the momentum $\vec{p} := E\vec{v}_c^{\text{cla}}$ it forms a 4-vector

$-E^2 + \|\vec{p}\|^2$

$$p := (E, \vec{p}) = \frac{1}{\sqrt{1 - \|\vec{v}_c^{\text{cla}}\|^2}} m_0 \begin{pmatrix} 1 \\ \vec{v}_c^{\text{cla}} \end{pmatrix} = m_0 v_c.$$

$\langle\langle p, p \rangle\rangle = -m_0^2$

As the proportionality constant between $\vec{p}$ and $\vec{v}_c^{\text{cla}}$ is the (relativistic) mass, we obtain that the relativistic energy and relativistic mass coincide. The last displayed equation implies, in particular, $\langle\langle p, p \rangle\rangle = -m_0^2$, and thus we obtain the famous **energy-momentum relation** (recall $c = 1$).

$$E^2 = \|\vec{p}\|^2 + m_0^2.$$

$g : V^* \times V^* \to \mathbb{R}$

e^i gen. onb $g(e^i, e^i) = g(e_i, e_i) = \epsilon_i$ $\quad V^* \ni e^i : V \to \mathbb{R}$

1.5 Special relativity: Electromagnetism

$e^i(e_j) = \delta^i_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Definition 1.5.1. *Let V be an oriented real vector space, $n := \dim V$, with a non-degenerate symmetric form g (i. e., an inner product), and let $(e_1, \dots, e_n)$ be a positively oriented generalized orthonormal basis of signature $(m, n-m)$. Then let $(e^1, \dots, e^n)$ be the algebraically dual basis, which is a basis of V^* and satisfies $e^i = \epsilon_i g(e_i, \bullet)$. It is a positively oriented generalized orthonormal basis for V^*, which we equip with the dual inner product having the same signature and sign factors ϵ_i.*

We obtain a natural volume element on V given by

$$\mathrm{vol} = e^1 \wedge e^2 \wedge \dots \wedge e^n \in \bigwedge^n V^*.$$

$\Lambda^n V^*$

$\mathrm{vol} : \underbrace{V \times \dots \times V}_{n\text{-times}} \to \mathbb{R}$ alternating

For $k = 0, 1, \dots, n$ the space $\bigwedge^k V^*$ carries an inner product, denoted again by g such that

$$\Big(\underbrace{e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_k}}_{\Lambda^k V^* \ni e^I :=} \;\Big|\; 1 \le \underbrace{i_1 < i_2 \dots < i_k}_{I:=} \le n\Big)$$

is a generalized orthonormal basis of $\bigwedge^k V^*$ with $g(e^I, e^J) = \delta_{IJ}\epsilon_I$, where $\epsilon_{i_1 < i_2 \dots < i_k} = \epsilon_{i_1}\epsilon_{i_2} \dots \epsilon_{i_k}$. There is now a well-defined ismorphism

$$*: \bigwedge^k V^* \longrightarrow \bigwedge^{n-k} V^*$$

isom.

which satisfies

$\Lambda^k V^* \ni \;\; \to \;\; \Lambda^{n-k} V^*$

$\Lambda^n V^* \ni$

$$\underbrace{g(\alpha, \beta)}_{\in \mathbb{R}}\, \mathrm{vol} = \beta \wedge (*\alpha) \quad \forall \alpha, \beta \in \bigwedge^k V^*.$$

$\Lambda^n V^*$

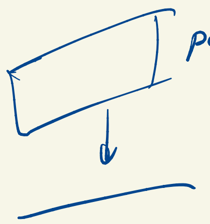
$g: V \times V \to \mathbb{R}$ inner product on V

$\rightsquigarrow \tilde{g}: V^* \otimes V^* \to \mathbb{R}$ inner product on V^*

$$\rightsquigarrow \tilde{\tilde{g}}: \underbrace{(V^* \otimes \dots \otimes V^*)}_{k\text{-times}} \otimes \underbrace{(V^* \otimes \dots \otimes V^*)}_{k\text{-times}} \to \mathbb{R}$$

$$(v_1 \otimes \dots \otimes v_k) \otimes (w_1 \otimes \dots \otimes w_k) \longmapsto \tilde{g}(v_1, w_1) \cdot \tilde{g}(v_2, w_2) \cdot \ldots \cdot \tilde{g}(v_k, w_k)$$

$v_1, \dots, v_k, w_1, \dots, w_k \in V^*$



$$V^* \otimes \dots \otimes V^* =: (V^*)^{\otimes k}$$

$$\downarrow$$

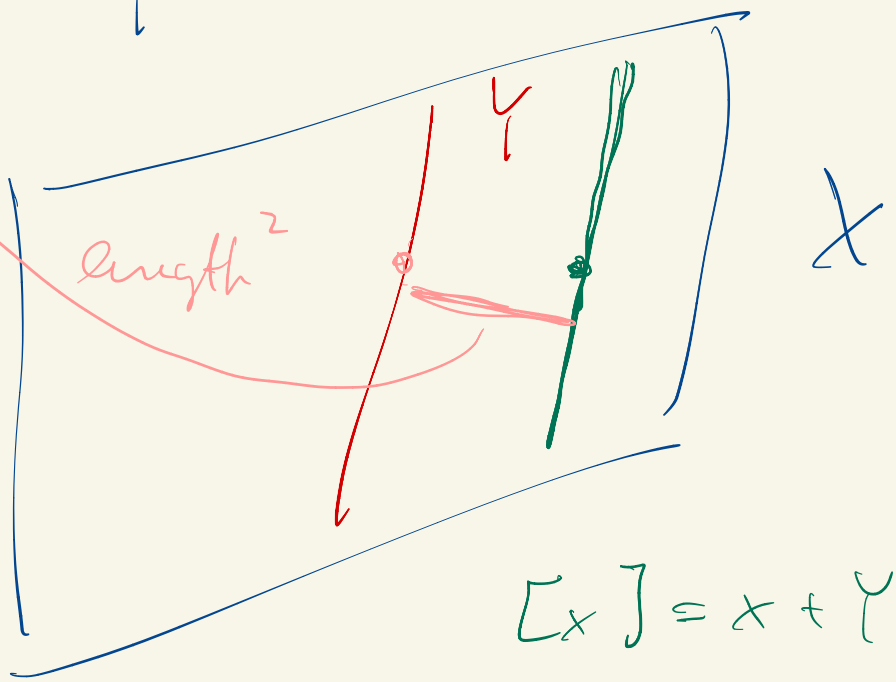
$$\Lambda^k V^* = (V^*)^{\otimes k} \Big/ \operatorname{span}\{ A \otimes \overbrace{(v \otimes w + w \otimes v) \otimes B}^{\in (V^*)^{\otimes k}} \}$$

$A \in (V^*)^{\ell}$ $\qquad B \in (V^*)^{k-\ell-2}$

If a vector space X carries a (positive definite) scalar product, $Y < X$, then X/Y has an induced s. prod

$$\langle [x],[x]\rangle = \min\left\{ \langle y,y\rangle \;\middle|\; x \ni y \in [x] \right\}$$

$\in X/Y$



$$[x] = x + Y \in X/Y$$

$\{1,\ldots,n\} \supset I = \{i_1 < \ldots < i_k\}$ $\qquad e^I$

Examples 1.5.2.

(a) For example let $I^c = \{j_1 < j_2 \ldots < j_{n-k}\} =: J$ be the set of indices not contained in I, ordered stricly increasing. Then

$$*(e^I) = \epsilon\epsilon_I\, e^J,$$

$\epsilon_{i_1<\ldots i_k} = \epsilon_{i_1}\cdots\epsilon_{i_k}$

where the sign ϵ_I is defined as above and where the sign ϵ is defined by $\mathrm{dvol} = \epsilon e^I \wedge e^J$, i. e., $\epsilon = 1$ if and only if $(e_{i_1},\ldots,e_{i_k},e_{j_1},\ldots,e_{j_{n-k}})$ is positively oriented. In particular $\quad \epsilon = -1$ if neg. or.

V signature $(m, n-m)$

$$*(1) = \mathrm{dvol}, \qquad *(\mathrm{dvol}) = (-1)^{n-m}.$$

For $\alpha \in \bigwedge^k V^*$ and $X \in V$ we have $\qquad X^\flat = g(X,\cdot) \in V^* = \Lambda^1 V^* \xrightarrow{*} \Lambda^{n-1}V^*$

$$*(*\alpha) = (-1)^{(n-k)k+(n-m)}\alpha, \qquad *(X^\flat) = \underbrace{\mathrm{dvol}(X,\bullet,\cdots,\bullet)}_{\Lambda^{n-1}V^*}.$$

(b) As a more concrete example, we equip $\mathbb{R}^{3,1}$ with the orientation, such that (e^0,e^1,e^2,e^3) is a positively oriented generalized orthonormal basis. Then

$$*1 = \mathrm{dvol}, \quad *e^0 = -e^1\wedge e^2\wedge e^3, \quad *e^1 = -e^0\wedge e^2\wedge e^3,$$

$$*e^2 = e^0\wedge e^1\wedge e^3, \quad *e^3 = -e^0\wedge e^1\wedge e^2, \quad *(e^0\wedge e^1) = -e^2\wedge e^3$$

$$*(e^0\wedge e^2) = e^1\wedge e^3, \quad *(e^0\wedge e^3) = -e^1\wedge e^2,$$

$$*(e^1\wedge e^2) = e^0\wedge e^3, \quad *(e^1\wedge e^3) = -e^0\wedge e^2, \quad *(e^2\wedge e^3) = e^0\wedge e^1,$$

$\Lambda^2 V^* \to \Lambda^2 V^*$

$$*(e^0\wedge e^1\wedge e^2) = -e^3, \quad *(e^0\wedge e^1\wedge e^3) = e^2, \quad *(e^0\wedge e^2\wedge e^3) = -e^1,$$

$$*(e^1\wedge e^2\wedge e^3) = -e^0, \quad *(\mathrm{dvol}) = -1.$$

$T^*V \qquad V^* \times V \xrightarrow{\pi_2} V$

We can apply this to forms, then e^i turns into dx^i.

V mfd $\qquad TV = V \times V$

$e^i \triangleq dx^i \in \Omega^1(V) \qquad e^I \in \Omega^k(M)$

$$\boxed{\begin{aligned} \mathrm{d}x^0 \wedge \mathrm{d}x^1 &\mapsto -\mathrm{d}x^2 \wedge \mathrm{d}x^3 && (1.5.1)\\ \mathrm{d}x^{\sigma(0)} \wedge \mathrm{d}x^{\sigma(1)} &\mapsto \varepsilon_\sigma \operatorname{sgn}(\sigma)\, \mathrm{d}x^{\sigma(2)} \wedge \mathrm{d}x^{\sigma(3)} && (1.5.2) \end{aligned}}$$

for $\sigma \in \operatorname{Perm}\big(\{0,1,2,3\}\big)$ and $\varepsilon_\sigma = \begin{cases} 1 & 0 \notin \{\sigma(0), \sigma(1)\} \\ -1 & 0 \in \{\sigma(0), \sigma(1)\} \end{cases}$.

We now choose a gauge such that several physical constants (in physics notation the speed of light c, the electric constant ϵ_0, the magnetic constant μ_0) are equal to 1. In experimental physics the Maxwell equations are determined as follows:

Maxwell equations.

axial vector

We have

- an electrical field $E\colon \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \qquad (t,x) \mapsto E(t,x)$.
- a magnetic field $B\colon \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \qquad (t,x) \mapsto B(t,x)$.
- the (electrical) charge (density) $\rho\colon \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$.
- the electrical current (density) $\vec{j}\colon \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

$\Lambda^2 \mathbb{R}^3 \cong \mathbb{R}^3$

They satisfy:

(i) $\operatorname{div}^{\mathbb{R}^3} E = \rho$. The **Gauß law**.

(ii) $\operatorname{div}^{\mathbb{R}^3}(B) = 0$. The **Gauß law for magnetic fields**.

(iii) $\operatorname{rot}(E) = -\frac{\partial B}{\partial t}$. The **Faraday law**.

(iv) $\operatorname{rot}(B) = \vec{j} + \frac{\partial E}{\partial t}$. The **Ampère–Maxwell law**.

$E = (E^1, E^2, E^3)^T$

$B = (B^1, \ldots)$

$\{$Two time dependent v.f. E, B on $\mathbb{R}^3\}$ $\overset{1:1}{\longleftrightarrow}$ $\Omega^2(\mathbb{R}^{3,1})$

We write

$$(F_{\mu\nu})_{\mu\nu} := \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \leftarrow dt, \qquad F := \sum_{0\le\mu<\nu\le 3} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{1.5.3}$$

$\uparrow dt$

in other words

$$F = E^\flat \wedge \mathrm{d}t + *(B^\flat \wedge \mathrm{d}t)$$

We calculate

$$\begin{aligned} \mathrm{d}F &= \mathrm{div}^{\mathbb{R}^3} B \cdot \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &\quad - \underbrace{\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3}_{\mathrm{dvol}} \left(\left(\frac{\partial B}{\partial t} + \mathrm{rot}(E) \right), \bullet, \bullet, \bullet \right). \end{aligned}$$

As a consequence F is closed if and only if (ii) and (iii) hold.

$$*F = -B^\flat \wedge \mathrm{d}t + *(E^\flat \wedge \mathrm{d}t)$$

$$\begin{aligned} \mathrm{d}(*F) &= (\mathrm{div}^{\mathbb{R}^3} E) \cdot \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &\quad + \mathrm{dvol}\left(\left(-\frac{\partial E}{\partial t} + \mathrm{rot}(B) \right), \bullet, \bullet, \bullet \right) \\ &= \mathrm{dvol}\left(\left((\mathrm{div}^{\mathbb{R}^3} E)\frac{\partial}{\partial t} - \frac{\partial E}{\partial t} + \mathrm{rot}(B) \right), \bullet, \bullet, \bullet \right) \\ &= *\left(\left((\mathrm{div}^{\mathbb{R}^3} E)\frac{\partial}{\partial t} - \frac{\partial E}{\partial t} + \mathrm{rot}(B) \right)^\flat \right). \end{aligned}$$

We define $J(x,t) := \begin{pmatrix} \rho(x,t) \\ \vec{j}(x,t) \end{pmatrix}$, then (i) and (iv) are equivalent to

$$*\mathrm{d}(*F) = J^\flat.$$

Result 1.5.3. *We have written the electromagnetic field as* $F \in \Omega^2(V)$*, for a* $(3+1)$*-dimensional Minkowski space. The Maxwell-equations are then equivalent to*

$$\mathrm{d}F = 0, \quad *\mathrm{d}(*F) = J^\flat.$$

Pulling back with any isometry $\mathbb{R}^{3,1} \to V$ *and using* (1.5.3) *we obtain the classical way to write* E *and* B*. Any rule how the elctromagnetic field changes under change of the inertial system is subsumed into this picture.*

This way to present the Maxwell equations allows to apply the Poincaré lemma. Because of the Poincaré lemma there is $A \in \Omega^1(\mathbb{R}^{3,1})$, the **4-vector potential** such that $F = \mathrm{d}A$. This A is unique up to a closed 1-form, as $F = \mathrm{d}A = \mathrm{d}\tilde{A}$ implies $\mathrm{d}(A - \tilde{A}) = 0$, i. e., there is some $f \in \mathcal{C}^\infty(\mathbb{R}^{3,1})$ with $A - \tilde{A} = \mathrm{d}f$.

What we did above is only the first step. Further steps are:

- The above form for the Maxwell equations generalizes immediately to Lorentzian manifolds, i. e., to curved spacetimes.
- It is now easy to view the Maxwell equations as the Euler-Lagrange equation of an action functional in the perspective of Lagrangian mechanics.
- The above considerations lead to describing the electromagnetic field as the curvature of a bundle. This bundle can be enlarged to a larger bundle which then describes the unification of the electromagnetic force with the weak force to the electroweak force. By enlarging the bundle further we obtain the "standard model of particle physics" which combines in an even larger bundle the electroweak force with the strong force. All this can be done on curved spacetimes without additional effort.