Lemma 1.2.5

(Connected Components of $O(m,1)$)

The Lorentz group of signature $(m,1)$ with $m \geq 1$ has precisely 4 connected components

So far:

$$O(m,1) \xrightarrow{\det} \{\pm 1\}$$

$$O(m,1) \xrightarrow{t\text{-or}} \{\pm 1\}$$

$$\Downarrow$$

$$A = (a^i_j)_{\substack{i=0,\dots,m \\ j=0,\dots,m}}, \quad |a^0_0| \geq 1.$$

$$t\text{-or}(A) = \operatorname{sign}(a^0_0)$$

Goal

$$O(n,1) \xrightarrow{(\det,\ t\text{-}or)} \{\pm 1\} \times \{\pm 1\}$$

Maps connected components of $O(n,1)$ bijectively to $\{\pm 1\} \times \{\pm 1\}$.

Remains to show

$$SO_{\uparrow}(n,1) := \{ A \in O(n,1) \mid \det A = 1,\ t\text{-}or(A) = 1 \}$$

is connected

Start with some $A \in SO_{\uparrow}(n,1)$

$$A = \left(a_0 \ a_1 \ \cdots \ a_n \right)$$

$$\langle\langle a_0, a_0 \rangle\rangle = -1 \quad , \quad a_0^0 \geq 1$$

$(\text{tr}(A) = 1)$

$$\Rightarrow \quad a_0 = \begin{pmatrix} \cosh\alpha \\ (\sinh\alpha) v \end{pmatrix} \qquad v \in S^{m-1}$$

$\alpha \in \mathbb{R}$

Determine $B_1 \in SO(m)$

$B_1(v) = e_1$

Choose $B_t \in SO(m)$ $\quad B_0 = \mathbb{1}_m$

$$A'_t := \begin{pmatrix} 1 & 0 \\ 0 & B_t \end{pmatrix} A$$

$$A'_0 = A \quad , \quad A'_1 = \begin{pmatrix} \cosh\alpha & * \\ \sinh\alpha & \\ 0 & \\ 0 & \end{pmatrix}$$

Now:

$$\hat{A}(t) := \begin{pmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & \underline{1\!\!1}_{m-1} \end{pmatrix}$$

$$\circ A_1'$$

Note $\hat{A}(0) = A_1'$

$$\hat{A}(t) = \begin{pmatrix} \cosh(t+\alpha) & \\ \sinh(t+\alpha) & * \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix}$$

$$\hat{A}(-\alpha) = \begin{pmatrix} 1 & 0 \, 0 \cdots 0 \\ 0 & \\ \vdots & C \\ 0 & \end{pmatrix} \quad C \in SO(m)$$

Now

$C \rightsquigarrow \mathbb{1}_n$ in $SO(n)$