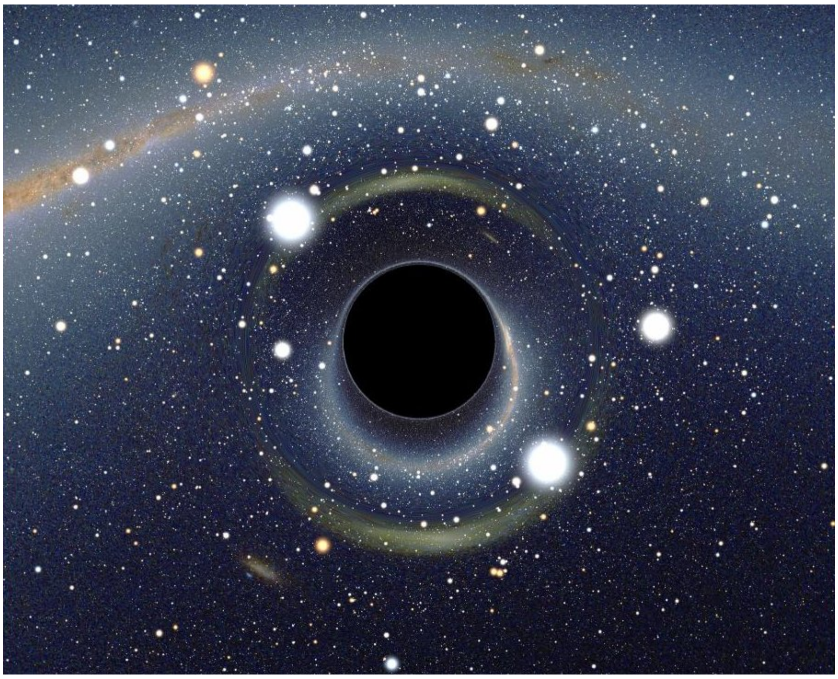
Differential Geometry II
Lorentzian Geometry
Lecture Notes



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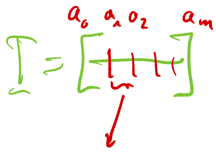
The group $\mathrm{SO}_{\uparrow}(m,1)$ defined above is called the **identity component of the Lorentz group**. We further define the **orthochronous Lorentz group** as

$$\mathrm{O}_{\uparrow}(m,1) := \{A \in \mathrm{O}(m,1) \mid \text{t-or}(A) = +1\}.$$

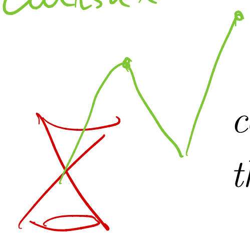
Definition 1.2.6. *A* **curve** *in a manifold M is a map $c\colon I \to M$, $t \mapsto c(t)$ where I is an interval. We then say that c is* **parametrized** *on I, and the argument t in $c(t)$ is called the* **parameter** *of c. We write $\dot c(t) \in T_{c(t)}M$ for the derivative of c in t. The curve is called C^k, $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$ if this holds as a map. Here ω stand for real-analytic. We say that a C^1-curve c is regular, if for all $t \in I$ we have $\dot c(t) \neq 0$.*



$c|_{[a_{j-1},a_j]} \in C^k$

A curve c, is called **piecewise C^k**, *if it is continuous, and if there are finitely many numbers $a_0 := \inf I < a_1 < \dots < a_\ell = \sup I$ such that $c|_{[a_{j-1},a_j]\cap I}$ is C^k for all $j = 1, \dots, \ell$. A piecewise C^1 curve $c : [a,b] \to M$ is regular if the decomposition $a_0 < a_1 < \dots < a_\ell$ can be chosen such that $c|_{[a_{j-1},a_j]\cap I}$ is a regular C^1-curve.*

non-causal



If (V,g) is a Minkowski space, then a piecewise C^1-curve $c : I \to V$ is called **timelike/lightlike/spacelike/nonspacelike**, *if for every $t \in I$ the vector $\dot c(t)$ is timelike/lightlike/spacelike/nonspacelike.*

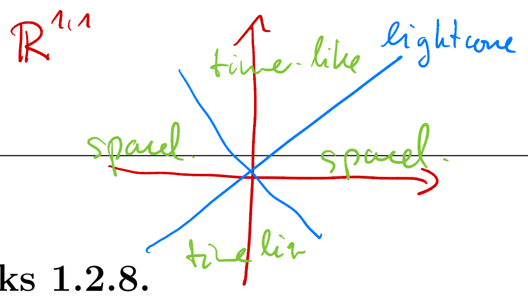
Note that the definition implies that any nonspacelike curve is regular.

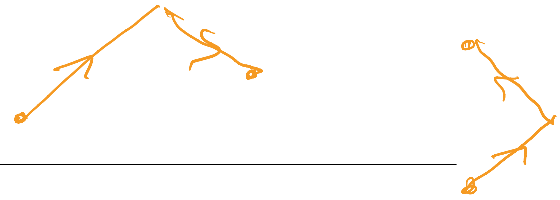
$a = \inf I$

Definition 1.2.7. *Let (V,g) be a Minkowski space. Assume that $c\colon I \to V$ is a (piecewise) C^1-curve. Then we define its* **length** *as* $b = \sup I$

$$\mathcal{L}(c) := \int_a^b \underbrace{\sqrt{|g(\dot c(t), \dot c(t))|}}_{\|\dot c(t)\| \text{ if } g \text{ were positiv definit}} \mathrm{d}t \in [0,\infty].$$

In case that c is a nonspacelike curve, then $\mathcal{L}(c)$ is also called the **proper time of the worldline** *c.*





Remarks 1.2.8.

(a) The length is independent under change of parametrization: if $\varphi : J \to I$ is a C^1-diffeomorphism, then $\mathcal{L}(c) = \mathcal{L}(c \circ \varphi)$.

(b) If $\dim V \geq 2$, then any two points can be joined by a piecewise C^1-curve of length 0. Thus the infimum lengths *cannot* be used to define a metric (in the sense of metric spaces, as we did for Riemannian manifolds.

$\mathcal{L}(c) = 0 \iff g(\dot{c}(t), \dot{c}(t)) = 0 \ \forall t \in I$

$\iff c$ light like

1.3 Special relativity: Why the Minkowski space?

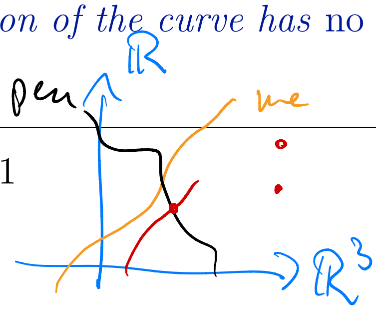
The content of this subsection is not of a mathematical nature. Instead, we want to provide arguments, inspired by physics, why it is reasonable to equip spacetime with the structure of a Minkowski space (as long as there are no curvature effects, i. e., "on a sufficiently small time and length scale"). More precisely, it has the structure of an affine Minkowski space with a time orientation, i. e., our spacetime is an affine space, whose associated vector space carries a metric of signature $(m, 1)$ with a time-orientation (see Definition 1.3.5).

In special relativity we join time $\mathbb{R}$ and our space $\mathbb{R}^3$ to a $(3+1)$-dimensional **spacetime** $\mathbb{R} \times \mathbb{R}^3$. Points in $\mathbb{R} \times \mathbb{R}^3$ will be called **events**. They consist of a time $x^0 := t \in \mathbb{R}$ and a (traditional) point $\vec{x} = (x^1, x^2, x^3)^\top$ in $\mathbb{R}^3$.

$\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$

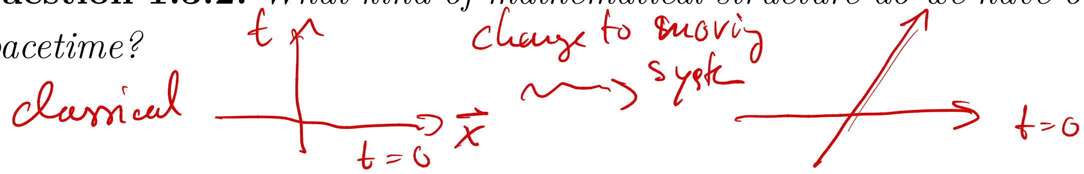
$t = x^0$

Physical Interpretation 1.3.1. *The physical interpretation of $x = (t, \vec{x})$ is that we are at a certain time x^0 at a traditional point $\vec{x}$. If an pointwise object moves in space (or if it remains at same place), it is alawys given by a curve $c : I \to \mathbb{R} \times \mathbb{R}^3$. If $(t, \vec{x}) = c(\tau)$ for some $\tau \in I$, this means that at time t the object will be at the position $\vec{x}$. The parametrization of the curve has* no *physical meaning.*



Up to here, we have just reformulated previously existing (Newtonian) physics.

Question 1.3.2. *What kind of mathematical structure do we have on spacetime?*

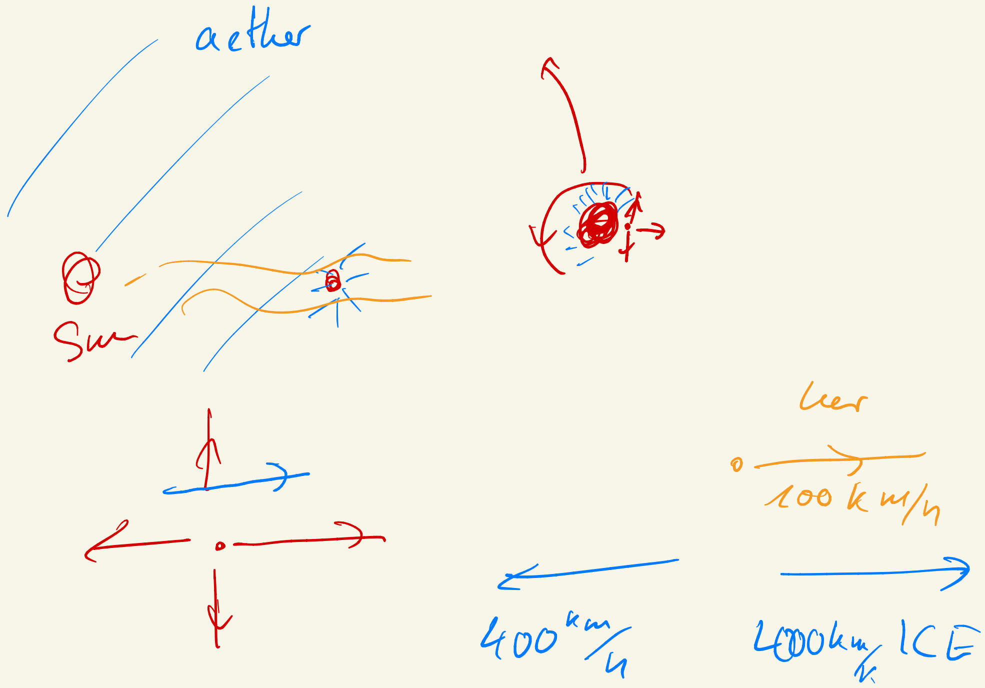


Physical Postulate 1.3.3 (Speed of light is constant)**.** *The speed of light in vacuum is constant. It does not depend on the status of motion of the observer (i.e. his speed and direction), and of the direction or polarization of the light, provided the observer is non-accelerated and non-rotating.*

(In fact the speed of light in modern gauge is $c = 299\,792\,458\, m/s = 1$.)

Note: a postulate is something that is claimed (by the physicists) and that is usually widely accepted in that community. Its logical role is similar to an axiom in mathematics: it is some assumption on which one builds a theory. In a physical theory the legitimization for a postulate may have different origins. They may come from an experiment, from an analogy to another physically relevant theory, from the fact that other assumptions would lead to contradictions or ill-defined objects, or just from esthetical considerations. In the case of the above postulate, the argument was derived from experiments, in particular from the Michelsohn-Morley experiment (1881).

A central idea of this experiment was that if you say "light moves at the speed x meters per seconds" then this always should depend on the status of motion of the observer. Assuming the postulates of classical Newtonian physics, one might argue as follows: If observer 1 observes that light moves with speed c_1 and if observer 2 moves with speed v_1 and in the same direction as the light, then observer 2 should


aether
Sun
Leer
100 km/h
400 km/h
4000 km/h ICE

mesasure (relative to him) that light travels with velocity $c_1 - v_1$. If he moves in the opposite direction, then he will measure $c_1 + v_1$. One can refine this argument: in any point $\vec{x}$ of space there is at most one velocity vector $\vec{v} \in \mathbb{R}^3$ such that an unaccelerated observer (observer 3) (moving along $\vec{x} + t\vec{v}$) will observe that the speed of light does not depend on its direction. So, if such an observer 3 exists (at $\hat{x}$), then in his measurements the speed of light does not depend on the direction, but all observers moving with some relative speed to him will measure that the speed of light depends on the direction. So he is in a very particular status of motion.

Towards the end of the 19th century physicists were able to measure quite precisely that in the system of motion described by a laboratory on the Earth's surface, the speed of light was nearly independent of the direction. The error of these measurements got essentially smaller than the speed of the Earth around the sub (about $30\,000 m/s$). So, we move (with sufficiently good approximation) in this particular direction. In view of Newtonian physics there were two solutions

(i) We are in absolute rest

(ii) There is a carrier medium, the "aether" in which light moves as a wave, similar to a water wave moves in water and sound in air. This aether moves with the Earth close to our planet's surface.

Possibility ((i)) seemed to be absurd: it would imply that we are in absolute rest, the sun, the planets and all other stars are moving around us following laws which are much more complicated than Newton's law. These would have meant abandonning the achievements of Copernicus, Galileo Galilei, Keppler, et. al.

So the dominant theory was the aether theory. The aether theory was also supported by the fact that scientist could not imagine, that a wave propagates without a carrier medium. The "aether" interpretation of the Michelsohn-Morley experiment then claimed that the aether was dragged by the Earth's surface. However, this was in contradiction to the Fizeau experiment which implied that the dragging effect of the aether could not be sufficiently strong. Also the dragging effect should have occured at other planets as well, potentially leading to optical turbulences which never where observed.

The final solution – given by Einstein's special relativity (1905) – was to abandon a postulate which noone questioned before, as it seemed so "obviously true".

We present this postulate in a stronger form (absolute time) and a weaker form (absolute time-order structure).

Abandoned Postulate (Absolute time)**.** *There is an absolute time. More precisely, there is an affine 1-dimensional space T and a map $t: S \to T$ from spacetime S to T which can be measured independently from the observer. (One says $t(\mathcal{E})$ is the time when the event $\mathcal{E}$ occured.)* $(t_2, \hat{x}_2)$ $t_2 - t_1$

time $(\hat{t}_1, \hat{\vec{x}}_1)$ $\mathcal{E}_1$ $\mathcal{E}_2$ $(t_2, \vec{x}_2)$ $(t_1, \vec{x}_1)$ obs obs $t_2 - t_1 \neq$

Abandoned Postulate (Absolute time-order structure and simultaneity)**.** *There is an absolute causal structure on spacetime, including a notion of simultaneity. More precisely, if two events $\mathcal{E}_1$ and $\mathcal{E}_2$ occur somewhere in spacetime it is possible to compare them, i. e., there is a well-defined answer - independent of the observer – to the question: did $\mathcal{E}_1$ happen earlier, at the same time or later than $\mathcal{E}_2$?*

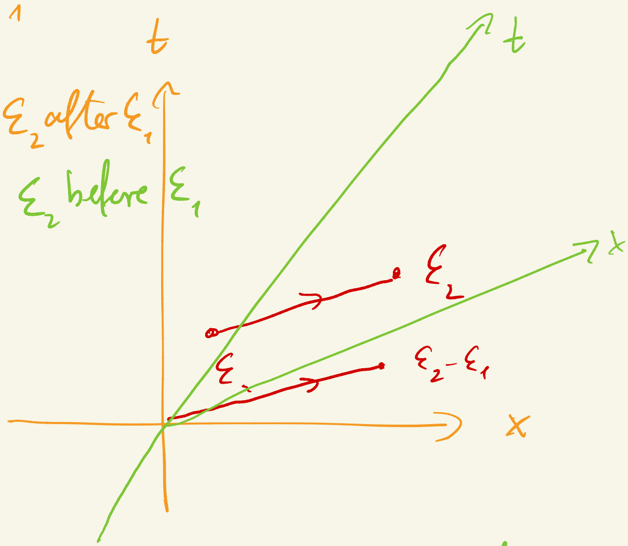
Obviously, the absolute time can be used to define an absolute time-order structure. I assume that still today, most people would not have any doubts that the Postulate of absolute time is a valid assumption.

However, this postulate had to be given up. Special relativity predicts that there is no absolute time, and even no absolute time-order

$\mathbb{R}^{1,1}$

obs 1: $\mathcal{E}_2$ after $\mathcal{E}_1$

obs 2: $\mathcal{E}_2$ before $\mathcal{E}_1$



$\mathcal{E}_2 - \mathcal{E}_1$ spacelike

Classical physics

$\mathcal{E}_1$ happened before $\mathcal{E}_2$

$\mathcal{E}_1$ " at the same time as $\mathcal{E}_2$

$\mathcal{E}_1$ happened after $\mathcal{E}_2$

$$t(\mathcal{E}_1) \begin{array}{c} < \\ = \\ > \end{array} t(\mathcal{E}_2)$$

Lorentz-transformed coordinate axis

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$$

structure: the temporal order of two events, measured by some observer, may depend on the motion of the observer. E.g. if we measure on Earth that an explosion in Chicago happened at the same time as a sun flare on the star Alpha Centauri, then an astronaut in a rocket moving with 10 percent of the speed of light from Earth towards Alpha Centauri will measure that the sun flare on Alpha Centauri happened much earlier, while an atronaut moving with similar speed from Alpha Centauri towards us will measure that the explosion in Chicago was earlier. The new effect is called **relativity of simultaneity** or – slightly different – **relativity of time**.

A further postulate, supported by experiments, was:

Physical Postulate 1.3.4 (Light travels along lines)**.** *Any (non-accelerating, non-rotating) observer will measure that light travels along straight lines (more precisely: affine lines).*

So let us fix to (non-accelerating, non-rotating) observers O_1 and O_2. Each observer is also supposed to fix an orthornormal frame of traditional space, to fix an origin in traditional space ($\hat{x} = 0$), to fix an origin of time $t = 0$. We also assume that they are able to measure the times and location for every event (sufficiently close to them) in our spacetime S. This defines a chart for each observer and finally leads to the structure of a 4-dimensional manifold. For simplicity we, ignore these local versus global effects, so we assume that observer O_i gets a bijective map $f_i : S \to \mathbb{R} \times \mathbb{R}^3$ which associates to an event the time and the coordinates he will measure. We obtain a bijective map $\varphi := f_2 \circ (f_1)^{-1} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ which we assume – as one does often in physics – to be smooth, and by inverting the role of O_1 and O_2, also the smoothness of φ^{-1} seems natural. Together with Postulates 1.3.3 and 1.3.4, we see that φ is a diffeomorphism that maps lightlike lines to lightlike lines. Here a lightlike line is one of the form

$$\{(t_0 + t, \vec{x}_0 + t\vec{x}) \mid t \in \mathbb{R}\}$$

for some $t_0 \in \mathbb{R}$, $\vec{x}_0 \in \mathbb{R}^3$, and $\vec{x} \in \mathbb{S}^2$. This is a severe mathematical obstruction, which has important consequences, that we will only sketch: using Exercise Sheet 2, Exercise 1?? and results about conformal groups for Minkowski space [15] (see A.1!!! for details), this implies that φ is conformal (in the sense of Minkowski spaces), i. e., there is a positive function $\rho : \mathbb{R}^{n,1} \to \mathbb{R}_{>0}$ such that

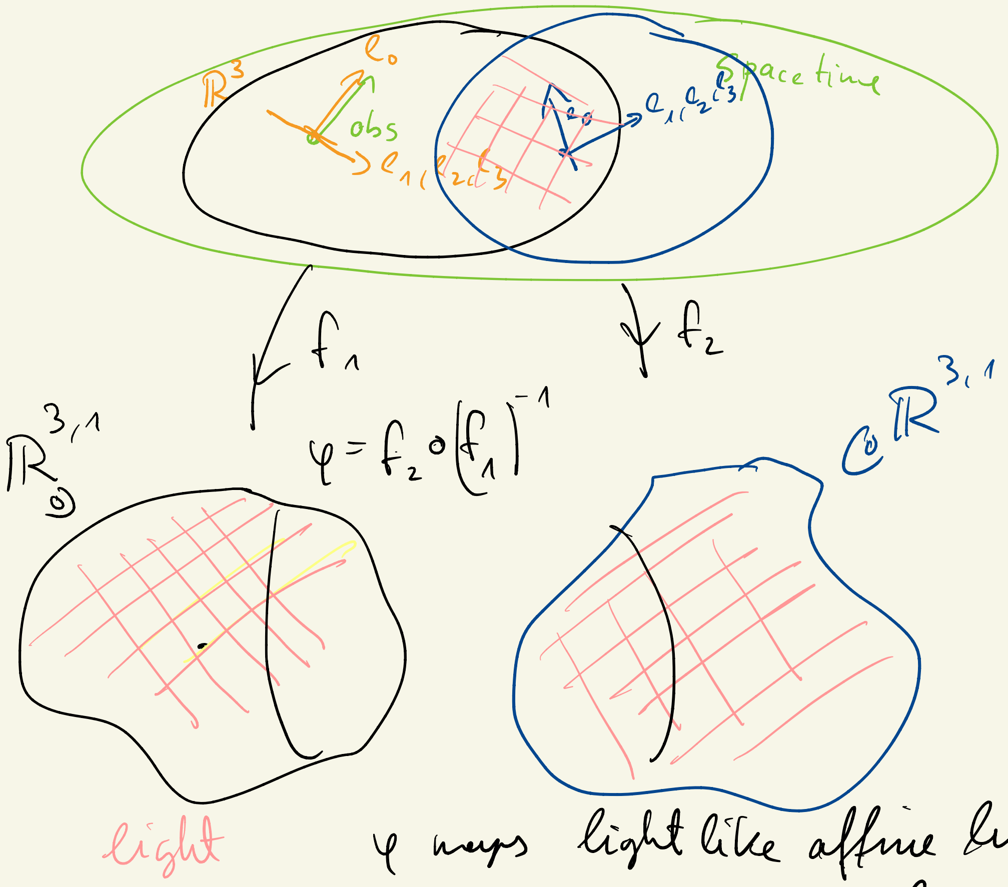
$$\forall p \in \mathbb{R}^{n,1}\, \forall x, y \in T_p\mathbb{R}^{n,1} : \langle\!\langle d_p\varphi(x), d_p\varphi(y)\rangle\!\rangle = \rho(p)\langle\!\langle x, y\rangle\!\rangle.$$

As the dimension S is larger than $2 = 1 + 1$ and as φ is defined everywhere in $\mathbb{R}\times\mathbb{R}^3$, this implies that φ has the form $x \mapsto cf(x)$ where c is a non-zero constants and where f is an element of the Poincaré group $\mathrm{O}(3, 1) \ltimes \mathbb{R}^{3,1}$. Assuming that the observer can distinguish past and future (see below), and determine a length scale by determining the frequency of the hyperfine structure of some atoms, we even see $\varphi \in \mathrm{O}_\uparrow(3, 1) \ltimes \mathbb{R}^{3,1}$.

On the other, if we postulate that observers can be accelerated sufficiently much, we can also argue that any Poincaré transformation in $\mathrm{O}_\uparrow(3, 1) \ltimes \mathbb{R}^{3,1}$ arises in this way. More precisely, for given $t_0 \in \mathbb{R}$, $\vec{x}_0, \vec{v} \in \mathbb{R}^3$ we may ask whether an observer no. 2 might follow the affine lines

$$\mathcal{L}_{t_0,\vec{x}_0,\vec{v}} := \{(t_0 + t, \vec{x}_0 + t\vec{v}) \mid t \in \mathbb{R}\},$$

expressed in coordinates of some fixed observer no. 1. There is no physical obstruction visible to the existence such an observer no. 2, as long as $\|\vec{v}\| < c = 1$, and it seems physically reasonable that No. 2 is able to change his coordinates of traditional space by any rigid motion of affine Euclidean space $\mathbb{R}^3$. This can be combined – but we will not



φ maps light like affine lines in $\mathbb{R}^{3,1}$ to lightlike affine lines

$$\varphi : \underset{\mathbb{R}^{3,1}}{\overset{U}{\cup\!\shortparallel}} \xrightarrow{1:1} \underset{\mathbb{R}^{3,1}}{\overset{V}{\cup\!\shortparallel}} \qquad d_p\varphi : \mathbb{R}^{3,1} \to \mathbb{R}^{3,1}$$

linear, preserves light like vectors.

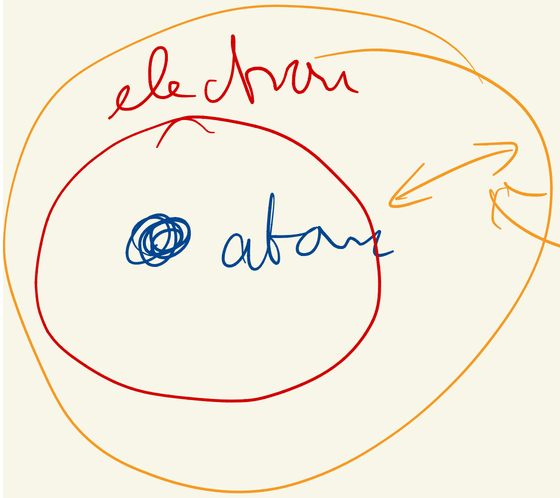
$\boxed{\text{Exercise 1 or 2 from Sheet no. 2}}$ Then $d_p\varphi = \underbrace{\rho(p)}_{\in \mathbb{R}_{\neq 0}} \cdot \underset{O(3,1)}{\overset{A}{\cap}}$

Cheat: $U = \mathbb{R}^{3,1}$ Poincaré group

Mathematical thm $\Longrightarrow$ $\exists\, c \in \mathbb{R}\setminus\{0\}$ $c\,\varphi \in \overbrace{\underbrace{O(m,1)}_{\text{Lorentz trafo}} \ltimes \underbrace{\mathbb{R}^{m,1}}_{\text{translations}}}^{\text{Poincaré group}}$

Additional Postulate

Observers can measure lengths



$\Delta E \rightsquigarrow$ light

... sec ... m

$\rightsquigarrow \varphi \in O(m,1) \ltimes \mathbb{R}^{m,1}$ does not depend on the observer

$\Longrightarrow \langle\langle\ ,\ \rangle\rangle$

The splitting

$$\mathbb{R}^{3,1} = \mathbb{R} \oplus \mathbb{R}^3$$

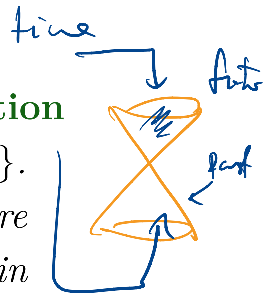
<u>not</u> preserved due to Poincaré trafo

splitting and the two factors $\mathbb{R}$ and $\mathbb{R}^3$ have "no physical reality" give details here – to argue that any element in $\mathrm{O}_{\uparrow}(3,1) \ltimes \mathbb{R}^{3,1}$ arises as a coordinate transformation from a fixed observer (no. 1) to some other observer.

Thus the structure of the lightcones is given by physical phenomena, and a time scale is given by standard gauging techniques. This information allows us to recapture the symmetric form $\langle\!\langle \bullet, \bullet \rangle\!\rangle$ on $\mathbb{R} \times \mathbb{R}^3$, and thus our space-times is modeled by the Minkowski space $\mathbb{R}^{3,1}$.

However, the splitting $\mathbb{R}^{3,1} = \mathbb{R} \oplus \mathbb{R}^3$ has *no* physical meaning. For any timelike future vector v there is an observer moving in the direction of v. This observer will interpret the orthogonal complement $v^{\perp}$ as the (traditional) space, and equipped with its Euclidean metric induced from $\langle\!\langle \bullet, \bullet \rangle\!\rangle$.

We have said above, that we expect that an observer should be able to distinguish past and future. We should spend some more thoughts on this, as it reflexts a further structure on our spacetime, namely a time-orientation.



Definition 1.3.5. *Let (V, g) be a Minkowski space. A* **time-orientation** *on (V, g) is a choice of a connected component of $\{x \in V \mid g(x,x) < 0\}$. Timelike vectors in this component and lightlike vectors in the closure of this component are called* **future-directed** *while causal vectors in (the closure of) the other component are called* **past-directed**.

Suppose we have to events $\mathcal{E}_1 = (t_1, \vec{x}_1)$ and $\mathcal{E}_2 = (t_2, \vec{x}_2)$ in $\mathbb{R}^{3,1}$. If $\mathcal{E}_2 - \mathcal{E}_1$ is causal and $t_2 > t_1$, then one can show that there is an event $\mathcal{E}_3 = (t_3, \vec{x}_3)$ with

- $t_3 \in [t_1, t_2]$
- $\mathcal{E}_3 - \mathcal{E}_1$ is zero or lightlike
- $\mathcal{E}_2 - \mathcal{E}_3$ is zero or lightlike

This means we can send a light signal from $\mathcal{E}_1$ to $\mathcal{E}_3$ reflect it there and send it then to $\mathcal{E}_2$. So one can transmit information from $\mathcal{E}_1$ to $\mathcal{E}_2$. On the other hand it would lead to logical contradictions if we could send an information to an event which happened earlier. Thus for $x, y \in \mathbb{R}^{3,1}$ with $x - y$ timelike we see that we can send information from x to y if and only if $y - x \in I_+(0)$. Thus the two open cones of time-like vectors can be distinguished by experiments, and this fact will be independent of the observer.

Physical Postulate 1.3.6 (Time-orientation)**.** *Our spacetime carries a time-orientation, detectable by experiments, which allows us to distinguish past and future.*

$$\mathrm{O}_{\uparrow}(m,1) \ltimes \mathbb{R}^{m,1}, \text{ where } \mathrm{O}_{\uparrow}(m,1) := \{A \in \mathrm{O}(m,1) \mid \text{t-or}(A) = 1\}.$$

As already mentioned above we expect that all (freely falling, non-rotating) observers shall experience the same physical laws. We will express and clarify this as a further postulate.

Physical Postulate 1.3.7 (Symmetry groups of special relativity)**.** *All physical laws of general relativity are invariant under the* **orthochronous Poincaré group** $\mathrm{O}_{\uparrow}(m,1) \ltimes \mathbb{R}^{m,1}$.

Above it remained open whether we can send information between events whose difference is spacelike. From our postulates we may deduce an answer

Physical Conclusion 1.3.8 (Speed of light is a universal speed limit, part 1)**.** *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are events in spacetime, with $\mathcal{E}_1 - \mathcal{E}_2 \neq 0$ spacelike, then it is impossible to send information from $\mathcal{E}_1$ to $\mathcal{E}_2$ or from $\mathcal{E}_2$ to $\mathcal{E}_1$. No observer may travel from $\mathcal{E}_1$ to $\mathcal{E}_2$ or vice versa.*

"Proof". We identify our spacetime with $\mathbb{R}^{m,1}$. If $\mathcal{E}_1 - \mathcal{E}_2 \neq 0$ is spacelike, then we can find $A \in \mathrm{SO}_\uparrow(m,1)$ such that the time component t_2 of $A(\mathcal{E}_2) = (t_2, \vec{x}_2)$ is larger than the time component t_1 of $A(\mathcal{E}_1) = (t_1, \vec{x}_1)$. Suppose we can send information from $\mathcal{E}_2$ to $\mathcal{E}_1$, then we can also send information from $(t_2, \vec{x}_2)$ to $(t_1, \vec{x}_1)$. We may determine a Poincaré transformation $P \in \mathrm{SO}_\uparrow(3,1) \ltimes \mathbb{R}^{3,1}$ such that $P((t_2, \vec{x}_2)) = (t_1, \vec{x}_1)$ and $P((t_1, \vec{x}_1) = (2t_1 - t_2, \vec{x}_2)$. Thus we may send information from $(t_1, \vec{x}_1)$ to $(2t_1 - t_2, \vec{x}_2)$, but transitivity this allows us to send it from $(t_2, \vec{x}_2)$ to $(2t_1 - t_2, \vec{x}_2)$. But because of $2t_1 - t_2 < t_2$ we obtain $(2t_1 - t_2, \vec{x}_2) - (t_2, \vec{x}_2) \in I_-(0)$, thus we could transmit information in the past. Any observer is able to transport information along his worldline, so he cannot travel between events which are spacelike to each others. $\square$

The above conclusion still does exclude particles or objects traveling faster than light, provided that they do not carry information. Such hypothetical particles, traveling faster than light are called **tachyons**, but they never been detected in confirmed experiments. So it is common consensus to exclude particles traveling faster than light.

1.4 Special relativity: Main ingredients

In the previous section we provided physical arguments why it is reasonable to assume in special relativity that our spacetime is modeled by a (3+1)-dimensional affine Minkowski with a time-orientation. This space will often be identified with $\mathbb{R}^{3,1}$ in the sequel, which amount to taking the perspective of some observer, in other words the choice of an **inertial system**. We discussed that this choice is unique up to the action of the orthochronous Poincaré group $\mathrm{O}_\uparrow(3,1) \ltimes \mathbb{R}^{3,1}$.

One now has to adapt the physical laws that we know from classical physics to the context of special relativity. One important claim is that for speeds essentially below the speed of light classical mechanics

$\mathbb{R}^3$ $\mathbb{R}$

is a good approximation for general relativity, a statement which could be made precise as a limit. This will lead to equations of motions and many other physical laws. Unfortunately, we will not be able to give a full derivation of these laws, as this would – as a first step – require a detailed study of classical mechanics, classical field theory and many more.

$\mathbb{R}^{3,1}$

Instead we will state some laws and draw some conclusions.

$\cup$

In special relativity we combine classical objects to relativistic objects, which are called "4-vectors" by physicists, for example:

classical scalar	name	classical vector	name	relativistic object	name
t	time	$\vec{x}$	position	$x = (t, \vec{x})$	event
E	energy	$\vec{p}$	momentum	$p = (E, \vec{p})$	(energy-)momentum 4-vector
ρ	electrical charge	$\vec{j}$	electrical current	$J = (\rho, \vec{j})$	electrical 4-current
ϕ	electric potential	$\vec{A}$	magnetic potential	$(\phi, \vec{A})$	electromagnetic potential

As an example we claim that all freely falling objects move along straight affine lines in $\mathbb{R}^{3,1}$, i. e., along worldlines

$$\mathbb{R} \ni \tau \in x(\tau) = x_0 + \tau v_0 \in \mathbb{R}^{3,1}$$

where $x_0 \in \mathbb{R}^{3,1}$ and v_0 is causal future-directed, and if the objects has non-zero rest mass m_0 we have the additional requirement that v_0 has to be timelike.

More generally, an arbitrary object is decribed by a worldline $\tau \mapsto x(\tau)$, a curve in spacetime, and two worldline will be considered identical if the arise from each other by a reparametrization. In the follow-

world line ~~non~~-spacelike

$c(t)$ future oriented