Differential Geometry II: Exercises

University of Regensburg, Summer term 2021
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Please hand in the exercises until **Tuesday, June 29**

Exercises Sheet no. 11

1. Exercise (4 points).
In the following M is always a time-oriented Lorentzian manifold.

a) For a subset $A \subseteq M$ show: $\mathrm{edge}(\overline{A}) \subseteq \mathrm{edge}(A)$ and $\mathrm{edge}(A) \setminus \mathrm{edge}(\overline{A}) \subseteq \overline{(\partial A) \setminus A}$, Give examples to show that both inclusions are proper inclusions in general.

b) Give an example of some connected M and a closed, non-empty spacelike hypersurface of M that is not achronal.

c) Give an example of a smooth hypersurface in $\mathbb{R}^{1,1}$ that is acausal, but not spacelike.

2. Exercise (4 points).

a) If V and W are two vector fields on M which are linearly independent in any point. Show that there is a Lorentzian metric g on M such that

i) V is timelike and W is lightlike

ii) V and W are lightlike

b) Construct a Lorentzian metric on $M := S^1 \times \mathbb{R}$ such that a $p \in M$ exists with $J^+(p)$ open and with $\overline{J^+(p)}$ compact.
Hint: a possible such Lorentzian metric has $S^1 \times \{0\}$ and $S^1 \times \{1\}$ has the images of two closed lightlike smooth curves.

3. Exercise (4 points).

a) Let $c : (a,b) \to M$ be a smooth lightlike curve in a Lorentzian manifold. Show that c is a pregeodesic if and only if there is no smooth variation $c_\bullet$ of c with compact support such that

$$\frac{\partial}{\partial s}\Big|_{s=0} g(\dot{c}_s(t), \dot{c}_s(t)) \leq 0$$

for all $t \in (a,b)$ and with the strict inequality for some $t \in (a,b)$.
Hint: Use methods similar as in Lemma (3.)2.29.

b) Let $f : M \to \mathbb{R}_{>0}$ be a smooth function and $\tilde{g} := f^2 g$. Use a) to show that a curve is a lightlike pregeodesic with respect to g if and only if it is a lightlike pregeodesic with respect to $\tilde{g}$.

4. Exercise (4 points).
Let (M,g) be the Schwarzschild spacetime with $M = \mathbb{R} \times ((2m)^{\frac{1}{n-2}}, \infty) \times S^{n-1}$ and

$$g(t,r,x) = -\left(1 - \frac{2m}{r^{n-2}}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{2m}{r^{n-2}}}\mathrm{d}r^2 + r^2 g_{S^{n-1}}(x)$$

and let $F: ((2m)^{\frac{1}{n-2}}, \infty) \to \mathbb{R}$ be a function whose derivative is $r \mapsto \left(1 - \frac{2m}{r^{n-2}}\right)^{-1}$.

a) Show that F is strictly monotonously increasing with $\lim_{r \to (2m)^{\frac{1}{n-2}}} F(r) = -\infty$ and $\lim_{r\to\infty} F(r) = \infty$. Conclude that F has a continuous inverse $F^{-1}: \mathbb{R} \to ((2m)^{\frac{1}{n-2}}, \infty)$.

b) Let now $\gamma: [a,b] \to M$, $\tau \mapsto (t(\tau), r(\tau), x(\tau))$ be a future-causal (piecewise C^1) curve and $\tau_0 \in [a,b]$. Prove that

$$F_1(t(\tau)) \leq r(\tau) \leq F_2(t(\tau)) \qquad \text{for all } \tau \geq \tau_0$$
$$F_2(t(\tau)) \leq r(\tau) \leq F_1(t(\tau)) \qquad \text{for all } \tau \leq \tau_0,$$

where $F_1(t) = F^{-1}(F(r(\tau_0)) + t(\tau_0) - t)$ and $F_2(t) = F^{-1}(F(r(\tau_0)) + t - t(\tau_0))$.

c) Show that (M,g) is globally hyperbolic.
Hint: You may use without further proof that in the definition of globally hyperbolic the strong causality condition may be equivalently replaced by the (non-strong) causality condition.