Differential Geometry II: Exercises

University of Regensburg, Summer term 2021
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Please hand in the exercises until **Tuesday, June 22**

Exercises Sheet no. 10

1. Exercise (4 points).
Let M^2 be a Riemannian surface with Gauß curvature $K \geq 1$ and $c\colon [a,b] \to M$ be a geodesic of M parametrized by arc-length. Show that there is a conjugate point to $c(a)$ along c if $b - a \geq \pi$.
Hints: For a parallel vector field $N(t)$ along c, normal to $\dot{c}(t)$, determine a differential equation for f such that $J(t) = f(t)N(t)$ is a Jacobi field. Proceeding similarly as in Exercise 3 of sheet 9, show that $h(t) = f'(t)/f(t)$ satisfies $h(t) \leq \cot(t-a)$ for all $t \in (a,b)$ as long as there is no conjugate value in (a,b).

2. Exercise (6 points).
Let $c : [0,b] \to M$ be a smooth curve in a semi-Riemannian manifold M. Let $V, A \in \Gamma(c^*TM)$. We define the variation

$$\begin{aligned} c_\bullet : [0,b] \times (-\epsilon,\epsilon) &\to M, \\ (t,s) &\mapsto c_s(t) := \exp_{c(t)}\left(sV(t) + \frac{1}{2}s^2 A(t)\right) \end{aligned}$$

of c, and define

$$\begin{aligned} D : [0,b] \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) &\to TM, \\ (t,\sigma,\tau,\rho) &\mapsto \left(d_{\sigma V(t)+\tau^2 A(t)} \exp_{c(t)}\right)(V(t) + \rho A(t)) \end{aligned}$$

a) Calculate $\frac{\nabla}{d\sigma} D(t,0,0,0)$ *(Hint: geodesic equation)*, $\frac{\nabla}{d\tau} D(t,0,0,0)$, and $\frac{\nabla}{d\rho} D(t,0,0,0)$.

b) Show $\frac{\partial c_s}{\partial s}\big|_{s=0} = V$. Use the chain rule to calculate $\frac{\nabla}{ds}\big|_{s=0} \frac{\partial c_s(t)}{\partial s} = A(t)$.

Now we additionally assume that P a semi-Riemannian submanifold of P with vector-valued second fundamental form $\vec{\mathrm{II}}$, and assume $p := c(0) \in P$, $\dot{c}(0) \perp T_pP$, $V(0) \in T_pP$ and $A(0) = \vec{\mathrm{II}}_p\big(V(0), V(0)\big)$.

Show that there is a variation $c_s : [0,b] \to M$, $s \in (-\epsilon,\epsilon)$ of c with the following properties:

c) Show that there is a variation

$$\tilde{c}_\bullet : [0,b] \times (-\epsilon,\epsilon) \to M$$

of c which satisfies $\frac{\partial \tilde{c}_s}{\partial s}\big|_{s=0} = V$, $\frac{\nabla}{ds}\big|_{s=0} \frac{\partial \tilde{c}_s}{\partial s} = A$, and $s \mapsto \tilde{c}_s(0)$ is a geodesic in P.
(Hint: Show that in a chart around p, the variation $c_\bullet$ achieves this goal up to terms of order 3 in the Taylor expansion. Use a cut-off function to improve this.)

3. Exercise (6 points).
Prove that if $P \subset M$ is a submanifold of a Riemannian manifold M and $c : [a, b + \epsilon) \to M$ a unit speed geodesic amenating perpendicularly from P such that P has a focal point at $c(b)$, then for all $b < t < b + \epsilon$ the geodesic $c|_{[a,t]}$ is not a curve of minimal length between $c(a)$ and $c(t)$.

(Hint: For this exercise go through the subsection 3.5 of the lecture notes "Differential Geometry I" by Clara Löh. Adapt them to our setting, using the notation of the DG II lecture. You may use without a proof and without restating that the computations in the proof of Thm. 3.2.8 in DG I hold.) You may proceed as follows:

a) Compute the missing term, that turns the variation formula with fixed ends of Thm. 3.5.2 into one with loose ends.

b) Follow the strategy of proof for Thm. 3.5.15 and use the definition of focal points in order to get rid of the interfering terms.