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Please hand in the exercises until **Tuesday, June 15**

Exercises Sheet no. 9

1. Exercise (4 points).
Let $P \subseteq M$ be a semi-Riemannian submanifold, and $c\colon [a,b] \to M$ a geodesic of M with $c(a) \in P$ and $\dot{c}(a) \in N_{c(a)}P$.

a) Show that if J is a Jacobi field along c that exhibits $c(b)$ as a focal point of P, then $J \perp \dot{c}$ everywhere.

Now consider the following statement: P does not have a focal point along c at b if and only if for all $v_1 \in T_{c(a)}M$ and $v_2 \in T_{c(b)}M$ with $v_1 \perp \dot{c}(a)$ and $v_2 \perp \dot{c}(b)$ there exists a Jacobi field J along c with $\pi^{\mathrm{nor}}(J(a)) = \pi^{\mathrm{nor}}(v_1)$, $\pi^{\tan}(\frac{\nabla J}{\mathrm{d}t}(a)) = \pi^{\tan}(v_1) - S_{\dot{c}(a)}\pi^{\tan}(J(a))$ and $J(b) = v_2$.

b) Formulate it in the following special cases:

   i) $P = \{c(a)\}$ is a point.

   ii) P is a hypersurface of M with $\mathrm{II}_{c(a)} = 0$.

   Hint: You should be able to get rid of all tangential and normal projections.

c) Prove the statement.

2. Exercise (4 points).
Let P be a k-dimensional submanifold of $\mathbb{R}^n$, equipped with the Euclidean metric, viewed as a Riemannian manifold. Identifying $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ the curves

$$t \mapsto c_{p,V}(t) := p + tV, \quad V \in N_pP$$

describe the geodesic intersecting P orthogonally. For any $V \in N_pP$ we define $S_V \in \mathrm{End}(T_pP)$ by the formula

$$\langle S_V(X), Y\rangle = \langle \vec{\mathrm{II}}_p(X,Y), V\rangle \quad \forall X, Y \in T_pP.$$

a) Show that P has a focal point along $c_{p,V}$ at $t \neq 0$ if and only if $1/t$ is an eigenvalue of S_V. The order of the focal point is then equal to the multiplicity of the eigenvalue.

b) Show that P has a focal point along $c_{p,V}$ at $t \neq 0$ if and only if the map $E : NP \to \mathbb{R}^n$, $N_pM \ni X \mapsto p + X$ has a non-regular point at $tV \in N_pM$. (Here non-regular means, that $d_{(p,tV)}E$ is not surjective.)

3. Exercise (4 points).
We assume that for $\kappa_0 \in \mathbb{R}_{>0}$ one of the following conditions holds

(i) (M, g) is a Riemannian manifold with sectional curvature $\sec \leq -\kappa_0$ and $c : \mathbb{R} \to M$ is a geodesic parametrized by arclength.

(ii) (M, g) is a Lorentzian manifold with sectional curvature $\sec \geq \kappa_0$ and $c : \mathbb{R} \to M$ is a timelike geodesic parametrized by proper time.

In both cases, let J we a Jacobi field along c orthogonal to c.

a) Show that
$$g\Big(J, \frac{\nabla^2}{dt^2} J\Big) \geq \kappa_0 g(J, J).$$

b) Set $f(t) := \sqrt{g\big(J(t), J(t)\big)}$. Show $f''(t) \geq \kappa_0 f(t)$. Conclude, if $f \not\equiv 0$, then it either attains its minimum in at most one $t_0 \in \mathbb{R}$ or is strictly monotonous.

c) Suppose $f'(t_1) > 0$.[1] Show that $h\colon (t_1, \infty) \to \mathbb{R}$, $h(t) = f'(t)/f(t)$ is well-defined and satisfies $h'(t) \geq \kappa_0 - h(t)^2$.

d) Show that for $F(t) := \cosh\big(\sqrt{\kappa_0}(t - t_1)\big)$ the function $H(t) := F'(t)/F(t)$ satisfies $H'(t) = \kappa_0 - H(t)^2$. Conclude $h(t) \geq H(t)$ for $t \geq t_1$.
(Hint: Lemma D.1 in Appendix D of the the partial script of the lecture might be helpful.)

e) Conclude that under the conditions of c), there is a constant $C > 0$ such that
$$g\big(J(t), J(t)\big) \geq C \cosh^2\big(\sqrt{\kappa_0}(t - t_1)\big), \quad \forall t \geq t_1.$$

[1] Obviously this can always be achieved at some $t_1 \in \mathbb{R}$, if we allow to exchange t by $-t$.