Differential Geometry II: Exercises

University of Regensburg, Summer term 2021
Prof. Dr. Bernd Ammann, Jonathan Glöckle, Roman Schießl
Please hand in the exercises until **Tuesday, May 18**

Exercises Sheet no. 5

1. Exercise (4 points).
Let (M, g) be a semi-Riemannian manifold and let $\gamma : (a, b) \to M$ be a geodesic. Let K be a Killing vector field which is defined as a vector field satisfying $\mathcal{L}_K g = 0$.

a) Show that $\tau \mapsto g(\dot\gamma(\tau), K|_{\gamma(\tau)})$ is constant. *Hint: Exercise 1 on sheet no. 4*

b) Consider the manifold $M = (a, b) \times S^{n-1}$ equipped with the warped product metric

$$g = dt \otimes dt + w(t)^2 g_{\mathrm{sph}},$$

where g_{sph} is the standard metric on S^{n-1} and $w > 0$ is smooth. We write $\gamma(\tau) = (\gamma^1(\tau), \sigma(\tau))$ with $\sigma(\tau) \in S^{n-1}$. Show that for any skew-symmetric $(n \times n)$-matrix A the expression

$$w\big(\gamma^1(\tau)\big)^2 \big\langle A\sigma(\tau), \dot\sigma(\tau)\big\rangle$$

does not depend on τ.

2. Exercise (4 points).
Consider $\mathbb{R}^2 \setminus \{0\} \ni (u, v)$ with the symmetric $(0, 2)$-tensor

$$g(u, v) = \frac{1}{u^2 + v^2}(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u).$$

a) Show that g defines a Lorentzian metric.

b) Argue that g induces a Lorentzian metric $\hat g$ on the quotient $M := (\mathbb{R}^2 \setminus \{0\})/((u, v) \sim (2u, 2v))$ and that M is diffeomorphic to a 2-torus.

c) Determine the geodesic $\gamma : I \to M$ with $\gamma(0) = [(1, 0)]$ and $\dot\gamma(0) = \frac{\partial}{\partial u}_{|\gamma(0)}$ on its maximal domain of definition I.

d) Conclude that $(M, \hat g)$ is compact but not geodesically complete, i. e. not every geodesic may be extended so that is defined on all of $\mathbb{R}$.

3. Exercise (4 points).
Verify that the *Schwarzschild metric*

$$g(t, r, x) = -\left(1 - \frac{2m}{r^{n-2}}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{2m}{r^{n-2}}}\mathrm{d}r^2 + r^2 g_{S^{n-1}}(x)$$

on $\mathbb{R} \times ((2m)^{1/(n-2)}, \infty) \times S^{n-1}$ is a vacuum solution of the Einstein equations.
Hint: You may apply the Ricci curvature formulae for warped products first to $P := \mathbb{R} \times (2m, \infty)$ (for which you thereby obtain $\mathrm{ric}^P = m(n-2)(n-1)r^{-n} g^P$) and afterwards to $P \times S^{n-1}$.

4. Exercise (4 points).
Let (M, g) be a $3+1$-dimensional Lorentzian manifold and $F \in \Omega^2(M)$ a closed 2-form. Interpreting F as electro-magnetic field, the electro-magnetic energy-momentum tensor may be defined by

$$T(X,Y) = \sum_{i=0}^{n} \epsilon_i F(e_i, X) F(e_i, Y) - \frac{1}{4} \sum_{i,j=0}^{n} \epsilon_i \epsilon_j F(e_i, e_j)^2 g(X,Y)$$

for $X, Y \in T_pM$, where $e_0, \dots, e_n$ is a generalized orthonormal basis of T_pM. Show the following:

a) T is symmetric and trace-free.

b) If there are no sources, i. e. if $J = \star \mathrm{d}(\star F) = 0$, then T is divergence-free.
Hints: You may use that $\mathrm{d}F(X,Y,Z) = \nabla F(X,Y,Z) + \nabla F(Y,Z,X) + \nabla F(Z,X,Y)$ and that $\star \mathrm{d}(\star F)(X) = -\sum_{i=0}^{n} \epsilon_i \nabla F(e_i, e_i, X)$.

c) For lightlike vectors $V, W \in T_pM$, $p \in M$, with $g(V,W) = -1$, one has $T(V,V) \geq 0$ and $T(V,W) \geq 0$.
Hint: Express F in a basis (V, W, X, Y) of T_pM, with respect to which g_p is given by the matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

d) T satisfies the dominant and the strong engery condition.