Differential Geometry II: Exercises

University of Regensburg, Summer term 2021
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Please hand in the exercises until **Tuesday, May 11**

Exercises Sheet no. 4

1. Exercise (4 points).
Let (M,g) be a semi-Riemannian manifold and $\{f_t : M \to M\}_{t\in I}$ ($0 \in I$ an inverval) a family of isometries that depends smoothly on t and for which $f_0 = \mathrm{id}$. Show that

$$\mathcal{L}_X g = 0 \text{ for } X|_p := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_t(p).$$

Now assume M to be compact (without boundary) and let X be any vector field. Prove that

a) X has a global flow $\phi : M \times \mathbb{R} \to M$,

b) the following statements are equivalent

   i) The flow of X consists of isometries.

   ii) $\mathcal{L}_X g = 0$.

   iii) For all points $p \in M$ and all vectors $Y \in T_pM$ the homomorphism

$$Y \mapsto \nabla_Y X$$

   with the Levi-Civita connection is skew-symmetric with respect to g_p, i.e.

$$g_p(\nabla_Y X, Z) + g_p(Y, \nabla_Z X) = 0.$$

2. Exercise: *Compare Exercise 2.5.11 in the Lecture Notes* (4 points).
We equip $M = B \times F$ with a warped product metric g_w for a warping function $w : B \to \mathbb{R}_{>0}$, and we use the notation of the lecture notes. Show that for vertical vector fields U, V, W we have

$$R(U,V)W = \hat{R}(U,V)W - \frac{\check{g}(\check{\mathrm{grad}} w, \check{\mathrm{grad}} w)}{w^2}\big(\hat{g}(V,W)U - \hat{g}(U,W)V\big).$$

Hints: you have to consider both the horizontal and the vertical part of this equation. Consider the subsection on homotheties.

3. Exercise (4 points).
Let $p : V \to M$ be a smooth vector bundle and $f : N \to M$ be a smooth map, then the pullback bundle $f^*V \to N$ with $f^*V := \{(v,p) \mid p \in N, v \in V|_{f(p)}\}$ is a smooth vector bundle. Let now ∇^V be a connection on V. Denote by $F : f^*V \to V$ the map covering f:

$$F : (v,p) \mapsto v$$

and define the pullback

$$F^*X = (X \circ f, \bullet) : p \mapsto (X|_{f(p)}, p) \text{ for } X \in \Gamma(V).$$

Show the following:

a) There is a unique connection ∇^{f^*V} on f^*V such that

$$\nabla^{f^*V}_X F^*Y = F^*\left(\nabla^V_{df(X)}Y\right) \text{ for all } X \in \mathfrak{X}(N), Y \in \Gamma(V).$$

b) If $V = TM$, g a semi-Riemannian metric on M and ∇^V its Levi-Cività connection, then for the induced bilinear form $\tilde{g}$ on the fibres of f^*TM it holds that

$$\nabla^{f^*TM}\tilde{g} = 0,$$

where the connection on tensor fields is defined as in Def. 2.1.14 with the Levi-Cività connection replaced by ∇^{f^*TM} and that with a chart $\varphi : U \to V$ of N

$$\nabla^{f^*TM}_{\frac{\partial}{\partial\varphi^i}}\left(df\left(\frac{\partial}{\partial\varphi^j}\right),\bullet\right) = \nabla^{f^*TM}_{\frac{\partial}{\partial\varphi^j}}\left(df\left(\frac{\partial}{\partial\varphi^i}\right),\bullet\right).$$

c) The curvature tensors are related by

$$R^{f^*V}(X,Y)v = \left(R^V(d_pf(X), d_pf(Y))F(v), p\right)$$

for $X, Y \in T_pN$ and $v \in (f^*V)|_p = V|_{f(p)} \times \{p\}$.

4. Exercise (4 points).
Let $(\bar{M}, \bar{g})$ be a flat semi-Riemannian manifold and M a semi-Riemannian submanifold of $\bar{M}$ of dimension m with induced metric g. For a basis $(b_1, \ldots, b_m)$ of T_pM satisfying $g(b_i, b_j) = \delta_{ij}\,\varepsilon_i \in \{-1, 1\}$ define the mean curvature vector $\vec{H}_p := \sum_{i=1}^m \varepsilon_i\,\vec{\mathbb{I}}(b_i, b_i)$.

a) Show that $\vec{H}_p$ is well-defined.

b) With ric the Ricci curvature of (M, g) show that for all $X, Y \in T_pM$ it is

$$\mathrm{ric(X,Y)} = \bar{g}(\vec{H}_p, \vec{\mathbb{I}}(X,Y)) - \sum_{i=1}^m \varepsilon_i\,\bar{g}(\vec{\mathbb{I}}(b_i, X), \vec{\mathbb{I}}(b_i, Y)).$$

c) Let now M be a hypersurface with (local) unit normal field ν and associated shape operator S. Show for the scalar curvature scal of (M, g) the equality

$$\bar{g}(\nu,\nu)\cdot \mathrm{scal} = (\mathrm{Tr}\,S)^2 - \mathrm{Tr}(S^2).$$