Differential Geometry II: Exercises

University of Regensburg, Summer term 2021
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Please hand in the exercises until **Tuesday, May 4**

Exercises Sheet no. 3

1. Exercise (4 points).
Determine the second fundamental form of the pseudosphere $\mathbb{S}^{n-1,k} := \left\{x \in \mathbb{R}^{n,k} \,\middle|\, \langle\!\langle x, x\rangle\!\rangle = 1\right\}$ and the pseudohyperbolic space $\mathbb{H}^{n,k-1} := \left\{x \in \mathbb{R}^{n,k} \,\middle|\, \langle\!\langle x, x\rangle\!\rangle = -1\right\}$ in $(\mathbb{R}^{n,k}, \langle\!\langle \bullet, \bullet\rangle\!\rangle)$ and calculate their respective Riemann curvature tensors.

2. Exercise (4 points).
A submanifold M of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called *totally geodesic* if for every geodesic $\gamma\colon(-\epsilon,\epsilon) \to \overline{M}$ of $(\overline{M}, \overline{g})$ with $\dot\gamma(0) \in TM$ there exists an $0 < \epsilon' \leq \epsilon$ such that $\gamma_{|(-\epsilon',\epsilon')}$ is contained in M. Show that the following are equivalent:

i) M is totally geodesic in $(\overline{M}, \overline{g})$.

ii) The induced second fundamental form $\mathrm{I\!I}$ vanishes on all of M.

iii) Every geodesic in M with respect to the metric g on M induced by $\overline{g}$ is also a geodesic of $(\overline{M}, \overline{g})$.

iv) For every geodesic $\gamma\colon(-\epsilon,\epsilon) \to \overline{M}$ of $(\overline{M}, \overline{g})$ with $\dot\gamma(0) \in TM$ there exists an $0 < \epsilon' \leq \epsilon$ such that $\gamma_{|(-\epsilon',\epsilon')}$ is a geodesic in M.

3. Exercise (4 points).
A semi-Riemannian manifold (M, g) is called *Einstein manifold* if there is a function $f \in C^\infty(M)$ such that $\mathrm{ric} = fg$.

a) Determine all Einstein manifolds of dimension $n \leq 2$.

b) Show that for any semi-Riemannian manifold $\frac{1}{2}\mathrm{d}\,\mathrm{scal} = \mathrm{div}\,\mathrm{ric}$ holds.
Hint: The *divergence* $\mathrm{div}\,K$ of a symmetric $(0,2)$-tensor $K \in \Gamma(T^*M \otimes T^*M)$ is defined as the metric trace of ∇K, i. e. by $\mathrm{div}\,K = \sum_{i=1}^n \epsilon_i\,(\nabla K)(e_i, e_i, \bullet)$ for a generalized local orthonormal frame $(e_1, \ldots, e_n)$.

c) Conclude that if an Einstein manifold (M, g) is of dimension $n \geq 3$ then the associated function f is locally constant and thus (M, g) has locally constant scalar curvature.

4. Exercise (4 points).
Let V be an n-dimensional real vector space. We define a map

$$\beta : \left(\bigwedge^2 V^*\right) \otimes \left(\bigwedge^2 V^*\right) \to \left(\bigwedge^3 V^*\right) \otimes V^*$$

by

$$\beta(R)(X,Y,Z,W) = R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W),$$

where $X, Y, Z, W \in V$. Show the following:

a) The map β is a well-defined linear map.

b) Some $R \in V^* \otimes V^* \otimes V^* \otimes V^*$ satisfies the pointwise symmetries of the curvature tensor (i. e. anti-symmetry in the first and second as well as the third and forth argument, interchange of pairs symmetry and first Bianchi-identity) if and only if R lies in the kernel of β.

c) The map β is surjective.
Hint: Choose a basis $(b_1, \dots, b_n)$ of V^* and consider first

$$\beta\left((b_i \wedge b_j) \otimes (b_k \wedge b_\ell)\right) \text{ for } \ell \in \{i,j\} \text{ and for } \ell \notin \{i,j,k\}$$

and afterwards

$$\beta\left(((b_k + b_\ell) \wedge b_i) \otimes (b_j \wedge (b_k + b_\ell))\right).$$

d) One has:

$$\dim \ker \beta = \binom{n}{2}^2 - n\binom{n}{3} = \frac{n^2(n^2-1)}{12}.$$