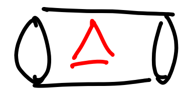
0. Motivation and Overview

Analysis II & IV Thy of submfds $M^n \subset \mathbb{R}^k$

 $\subset \mathbb{R}^3$

curvature

intrinsic: $R, K, Ric, \dots$

extrinsic: $H, \vec{\mathrm{II}}$

The intrinsic curvature is invariant under isometries

$$\begin{array}{ccc} M & \xrightarrow[f]{\text{isometries}} & M' \\ \downarrow & & \downarrow \\ \mathbb{R}^k & & \mathbb{R}^l \end{array} \qquad \Rightarrow \qquad f^* R^{M'} = R^M$$

$(\dim M = \dim M')$

$\rightsquigarrow$ (M, g) Riemannian mfd $g \in \Gamma(T^*M \otimes T^*M)$ symmetric, positive definit

Diffgeo 1 Curvature Jacobi fields

geodesics

R depends g_{ij}, $\partial_k g_{ij}$, $\partial_l \partial_k g_{ij}$.

Riem. geom
Lorentz $\Big\}\subset$ Semi-Riem. geom.

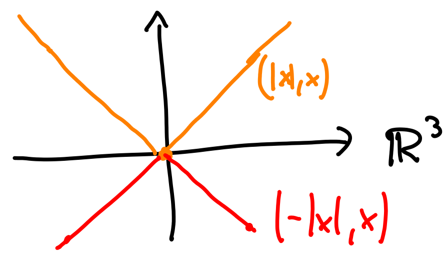
<u>Special relativity</u>: $\mathbb{R}^{1}\oplus\mathbb{R}^{3}=\mathbb{R}^{3,1}$ spacetime

(t, x) $\quad (x^0, x^1, \dots, x^3)\in\mathbb{R}^4$

$\;\; \| \quad\;\; \|$

$\mathbb{R} \quad x^0 \quad (x^1,x^2,x^3)$

$$\boxed{\begin{array}{l} c=1 \\ \cancel{3\cdot 10^{8}} \end{array}}$$



$t^2=|x|^2$ Cone in $\mathbb{R}^4=\mathbb{R}^{3,1}$

$$\left\langle \begin{pmatrix} t\\ x^1\\ x^2\\ x^3\end{pmatrix}, \begin{pmatrix} \tilde t\\ \tilde x^1\\ \tilde x^2\\ \tilde x^3\end{pmatrix}\right\rangle = -t\tilde t+\sum_{i=1}^{3} x^i\tilde x^i$$

Special relativity $\rightsquigarrow$ General relativity

Replace flat $\mathbb{R}^{3,1}$ $\rightsquigarrow$ Lorentzian mfd (M,g)

$\dim M=4$ mfd, $g\in\Gamma(\underbrace{T^*M\otimes T^*M}_{\text{symmetric}})$

signature $(3,1)$

Relation between (matter, fields) $\longleftrightarrow$ curvature of (M,g)

Einstein equations

Causality Past $\neq$ future

<u>Black hole</u> • Rotationally symmetric black holes, & stationary

Schwarzschild solution ≈ 1915

• S^1- symmetric black holes, black holes with

Kerr solution

• <u>Penrose singularity theorem</u>

He gave conditions under which black holes have to form, without symmetry assumptions, but assuming that the std laws of G.R. still hold.

Cosmology   Longterm of the space time

Robertson - Walker spacetimes, high degree of symmetry
Reduce Einstein equations to ordinary diff. equations.

$\rightsquigarrow$ big bang existed in past

Hawking Singularity theorem   Without symmetry assumptions, there are conditions such that an "intial singularity" has to exist.

Gravitational waves

$$\mathbb{R}^{3,1} \text{ flat space} \quad \overset{\text{Perturbations}}{\rightsquigarrow} \quad \text{gravitational waves}$$

kernel of the differential of the Einstein equations.

Einstein equation of the vacuum: $Ric = 0$.

$$Ric = 0 \not\Rightarrow R = 0$$

Yau's theorem   There are compact Riem. mfds (M, g) such $Ric = 0$, but $R \neq 0$.

$$M = \{X^4 + Y^4 + Z^4 + W^4 = 0\} \subset \underbrace{\mathbb{C}P^3}_{} \ni [X : Y : Z : W]$$

$X, Y, Z, W \in \mathbb{C}$

$\dim_{\mathbb{R}} M = 4$ $\dim_{\mathbb{R}} \mathbb{C}P^3 = 6$