Übungen zur Indextheorie

Universität Regensburg, Wintersemester 2016/17
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Exercise Sheet 11, due to 20.1.2017

Exercise 1

Show that the connected component of the identity $SO_0(n,1)$ of $O(n,1)$ is noncompact. Consequently, in the Lorentzian case we cannot proceed by the Haar measure construction of an invariant scalar product on the spinor module as in the Riemannian case. To that aim, let W be a Cl_n-module and

1. Describe the invariant scalar product: For E being the subgroup of Cl_n^x generated by an orthonormal basis $e_1, ..., e_n$, show that E contains $2 \cdot 2^n$ elements. Let $\langle \cdot, \cdot \rangle_0$ be any Hermitean bilinear form on W and define a Hermitean bilinear form $\langle \cdot, \cdot \rangle$ on W by

$$\langle v, w \rangle := \sum_{e \in E} \langle e \cdot v, e \cdot w \rangle_0.$$

   Show that this is an invariant scalar product and that Clifford multiplication is skew-symmetric w.r.t. $\langle \cdot, \cdot \rangle$.

2. Now for $n = k + l$ with $k, l \in \mathbb{N}$ define, for $\phi \in W$,

$$e_j \bullet \phi = \begin{cases} e_j \cdot \phi & \Leftrightarrow j \leq k \\ ie_j \cdot \phi & \Leftrightarrow j > k, \end{cases}$$

   and $\langle \phi, \psi \rangle_{k,l} := \langle e_{k+1} \cdot ... \cdot e_{k+l} \cdot \phi, \psi \rangle$. Show that this defines a $\mathrm{Cl}_{k,l}$-module structure with a $\mathrm{Spin}(k,l)$-invariant bilinear form for which Clifford multiplication with $v \in \mathbb{R}^{k,l} \in \mathrm{Cl}_{k,l}$ satisfies
   $\langle v \bullet \phi, \psi \rangle_{k,l} = i^{l-1} \langle \phi, v \bullet \psi \rangle_{k,l}$.

What is the signature of this invariant bilinear form in the Lorentzian case?

Exercise 2

Show that if equality holds in Friedrich's estimate on a Riemannian spin manifold (M,g) then (M,g) carries a Killing spinor, i.e. a spinor $\phi \in \Gamma(\Sigma M)$ satisfying $\nabla_X \phi = \alpha X \cdot \phi$ for all $X \in TM$.
Hint: Examine the connection $\tilde{\nabla}$ on the spinor bundle defined by $\tilde{\nabla}_X := \nabla_X - \lambda X \cdot$.

Exercise 3

Show that if a Riemannian spin manifold carries a Killing spinor then it is an Einstein manifold. If the spinor is parallel, then it is Ricci-flat.
Hint: Calculate $\sum_{i=1}^n e_i \cdot R(e_i, X)\phi$ in two ways.
Bonus question: Which weaker statement holds in the general semi-Riemannian case?