Übungen zur Indextheorie

Universität Regensburg, Wintersemester 2016/17
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Exercise Sheet 8, due to 15.12.2016

Exercise 1

Show Proposition 12.3: Let H, H' be separable Hilbert spaces. Prove that

1. $||A||^2_{\mathrm{HS}} = \langle A, A\rangle_{\mathrm{HS}}$ for all $A \in \mathcal{HS}(H, H')$ where $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$ is the scalar product given by $\langle A, B\rangle = \sum_{i,j} c_{ij}(A)\overline{c_{ij}(B)}$ for $A, B \in \mathcal{HS}(H, H')$ and c_{ij} given by two Hilbert basis as before,
2. $(\mathcal{HS}(H, H'), \langle\cdot,\cdot\rangle_{\mathrm{HS}})$ is a Hilbert space;
3. $||\cdot||_{\mathrm{HS}} \geq ||\cdot||_{H,H'}$,
4. Every Hilbert-Schmidt operator is compact,
5. For every $A \in \mathcal{HS}(H', H'')$ and every $B \in \mathcal{B}(H, H')$ we have $A \circ B \in \mathcal{HS}(H, H'')$.

Hint for the 4th item: Show first that both sides can be approximated in the respective norms by finite-rank operators.

Exercise 2

Show Proposition 12.5.: Prove that the sequences $\mathcal{TC}(H) \subset \mathcal{HS}(H) \subset \mathcal{K}(H) \subset \mathcal{B}(H)$ and $\ell^1 \subset \ell^2 \subset c_0 \subset \ell^\infty$ are connected by the respective restrictions of $\Phi_e : \mathcal{B}(H) \to \ell^\infty$ given by $\Phi_e(A) := (\sqrt{\langle Ae_k, Ae_k\rangle})_{k\in\mathbb{N}}$ (for a Hilbert basis $e = (e_n)_{n\in\mathbb{N}}$) and $\Psi_e : \ell^\infty \to \mathcal{B}(H)$ given by $\Psi_e((s_n)_{n\in\mathbb{N}}) = (v \mapsto \sum_{n\in\mathbb{N}} a_n\langle v, e_n\rangle e_n)$. Show that the restrictions of Ψ_e and Φ_e have the correct ranges, that the restrictions of Ψ_e are isometric monomorphisms of algebras for the respective metrics, and that $\Phi_e \circ \Psi_e$ corresponds to termwise taking the absolute value, in particular is the identity on nonnegative sequences.

Exercise 3

Show Proposition 8: The assignment $k \mapsto A_k$ as defined in the lecture is an isometry from $L^2(M \times M)$ to $\mathcal{HS}(L^2(M))$.